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On the periodic solutions of a non smooth dynamical system

ALAIN LEGER
Laboratoire de Mécanique et d'Acoustique, CNRS, Marseille, France
Université de Provence, Centre de Mathématiques et d'Informatique, Marseille, France
Auteur correspondant : leger@lma.cnrs-mrs.fr

ELAINE PRATT
Laboratoire de Mécanique et d'Acoustique, CNRS,
Université de Provence, Centre de Mathématiques et d'Informatique, Marseille, France

Abstract
A simple mass spring system with non regularized unilateral contact and friction conditions is submitted to an oscillating loading. The behaviour of this system is represented in the period-amplitude plane. We first observe the existence of stationary solutions despite the oscillating loading in a whole strip of this plane. For larger amplitudes of the loading, stationary solutions no longer exist and we prove the existence of sliding periodic solutions of which the multiplicity, the period and the smoothness depend on the period of the loading. We then compute the boundary of the range where these sliding periodic solutions exist, and we show that beyond this boundary, periodic solutions still exist but loose contact during a part of the period, so that impacts must be correctly taken into account.

Résumé
On étudie la dynamique d'un système simple masse-ressorts soumis à des conditions de contact unilatéral et de frottement non régularisés. Le système est soumis à une sollicitation périodique et l'on présente sa réponse dans le plan période-amplitude de l'excitation. On observe tout d'abord l'existence de solutions stationnaires dans toute une bande de ce plan, ce qui n'est pas habituel puisque les données sont oscillantes. Pour des amplitudes plus grandes de l'excitation les solutions stationnaires disparaissent et l'on voit apparaître des solutions périodiques glissantes, dont la multiplicité, la période et la régularité dépendent de la période de l'excitation. On calcule alors la frontière de la zone d'existence de ces solutions et l'on montre qu'au-delà de cette frontière il existe encore des solutions périodiques mais elles perdent le contact avec l'obstacle pendant une partie de leur période, ce qui en retour exige que les chocs soient traités correctement.

1. INTRODUCTION

Dynamical systems in continuum mechanics when contact and friction are involved constitute a difficult and open field of research. Some insight is however to be gained by studying small dimensional discrete systems. In this paper we explore the dynamics of a simple mass spring system (see figure 1) submitted to an oscillating force. This system was introduced in [6] in order to study the existence and the uniqueness of quasi-static solutions. After an investigation of the equilibrium states in [4], the stability of all the equilibria was given in [3] using classical concepts of stability. We study the trajectories $u(t) = (u_t(t), u_n(t))$ of the mass m when it is submitted to a force $F = (F_t, F_n)$ and the obstacle opposes an unknown reaction $R = (R_t, R_n)$ satisfying the contact conditions. We insist on the fact that the presence of the obstacle induces shocks, which produce discontinuities of the velocity. Moreover, as a constitutive law, we choose the simplest one from the physical point of view, which is that the mass cannot penetrate the obstacle, and that when the mass is in contact with the obstacle, it is submitted to Coulomb friction.

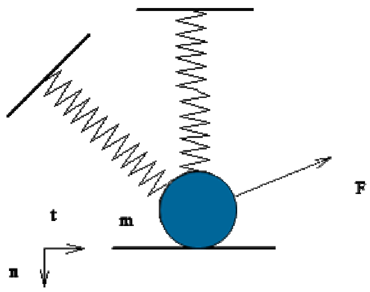

Figure 1: The simple mass-spring system

2. STATEMENT OF THE PROBLEM

We shall denote by m the mass of the particle, by $K = \begin{pmatrix} K_t & W \\ W & K_n \end{pmatrix}$ the stiffness of the system of springs and by μ the friction coefficient.

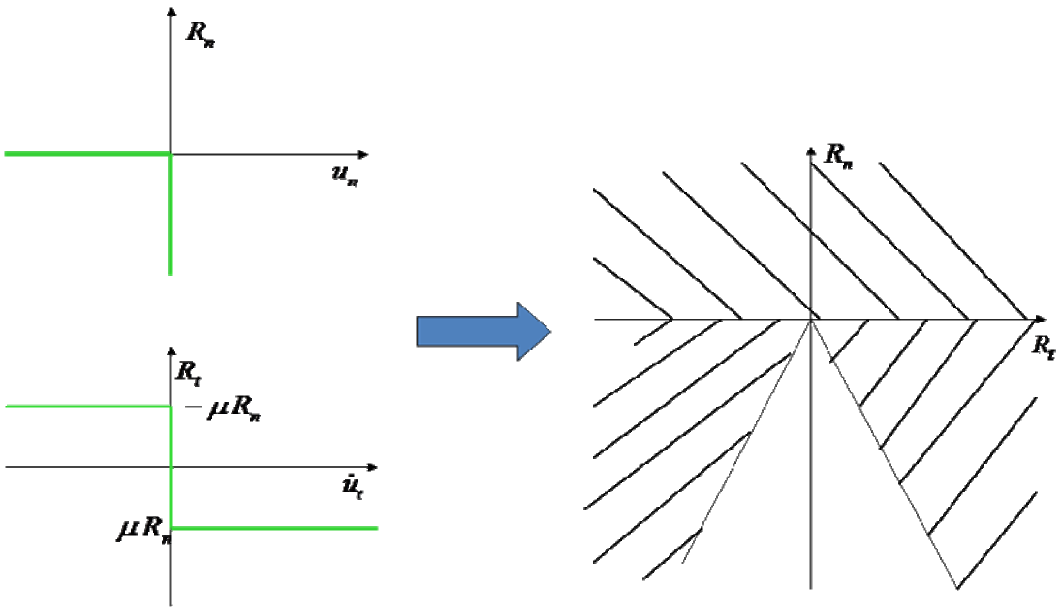

Figure 2: The graphs of the contact and friction conditions induce the admissible region for the reaction

As the mass cannot penetrate the obstacle, the normal reaction and the normal displacement must satisfy the following Signorini conditions represented in figure 2 by the graph on the top left hand side and written as:

$$u_n \leq 0, \quad R_n \leq 0, \quad u_n R_n = 0,$$

which means that the mass must remain above the obstacle, that the reaction can only push the mass upwards, that as long as the mass is strictly above the obstacle there is no reaction and that a non zero reaction is possible only when the mass is in contact with the obstacle.

In the same way the Coulomb friction conditions (represented in figure 2 by the graph on the lower left hand side) implies that the particle can slide only when the tangential component R_t of the reaction has attained one end of the interval $(\mu R_n, -\mu R_n)$ and when it slides its velocity and the tangential reaction are of opposite sign. By combining these two conditions we can see that the reaction must belong to the non shaded area of the $\{R_t, R_n\}$ plane represented on the right hand side of figure 2 because the Signorini conditions exclude the upper half of the $\{R_t, R_n\}$ plane and the Coulomb friction conditions imply that the reactions must belong to the cone, referred to as the Coulomb cone, represented in the lower half of the $\{R_t, R_n\}$ plane.

We can now give a precise mathematical formulation of the problem considered here. Note that as stated above there are two unknowns (the displacement and the reaction) but there is a single equation (the equation of motion) supplemented by a set of inequalities, so that the dynamical system will not be a classical one. In order to take impacts into account the velocity is a discontinuous function (more precisely it will be a function of bounded variation) so that the acceleration and the reaction are distributions and the sign condition on the reaction implies that these distributions are measures. The trajectories are the solutions to the following problem:

$$\begin{cases} Find \quad u \in \mathcal{MMA}([0,T],\mathrm{R}^2) \;\; and \;\; R \in \mathcal{M}([0,T],\mathrm{R}^2) \;\; such\,that: \\ m\ddot{u} = -Ku + F + R, \\ u(0) = u_0, \quad \dot{u}^+(0) = v_0, \\ u_n \le 0, \quad R_n \le 0, \quad u_n R_n = 0, \\ \forall \varphi \in C^0([0,T], IR) \qquad \int_0^T R_t(\varphi - \dot{u}_t^+) - \mu R_n(|\varphi| - |\dot{u}_t^+|) \le 0, \\ u_n(t) = 0 \Rightarrow \dot{u}_n^+(t) = -e\,\dot{u}_n^-(t), \quad e \in [0,1]. \end{cases} \tag{1}$$

The functional framework, first given in [10] and [7], involves $\mathcal{M}([0,T],\mathrm{R}^2)$ which denotes the set of measures defined on [*0,T*] with values in R^2 and $\mathcal{MMA}([0,T],\mathrm{R}^2)$ which denotes the set of continuous functions defined on [*0,T*] with values in R^2 the second derivative of which are measures.

3. FIRST RESULTS

Although at first glance one may be tempted to consider problem (1) as a simple Cauchy problem associated to an ordinary differential equation it turns out that a specific analysis is needed. We first recall the following results, on which the analysis is founded.

i. As presented in [4] the investigation of the set of equilibrium states involves two parameters depending on the set of data which are $K_t - \mu W$ and $K_t F_n - WF_t := A.$ Generically there exists either a single equilibrium state out of contact, or infinitely many equilibrium states in contact, but for some values of the data equilibrium states in contact may coexist with equilibrium states out of contact.
ii. Moreover, given initial data compatible with the obstacle, a trajectory exists as soon as the external force *F* is an integrable function of time on [*0*, *T*].
iii. The trajectory is not unique in general, so that the Cauchy problem is ill-posed even for an infinitely differentiable external force. A counter-example to uniqueness has been given in [10] in the case without friction in which two solutions to problem (1) are calculated explicitly for a given C^∞ force. Well-posedness is recovered only with analytical data (in fact piecewise analytical is sufficient). This was established in [2] in the case of Coulomb friction.
iv. In the case where the trajectory is unique, a time stepping algorithm has been built which converges towards the solution to problem (1). This algorithm is closely related to the functional framework and has been described extensively in [5].

As mentionned above, it has been shown that, given a constant external force such that the mass is in contact, problem (1) possesses in general infinitely many equilibrium solutions. It is actually easily obtained that the equilibrium equation, i.e. the right hand side of the equation of motion in problem (1) can be represented by a straight line in the $\{R_t, R_n\}$ plane so that the equilibrium solutions to problem (1) are given by the intersection of this line with the Coulomb cone.

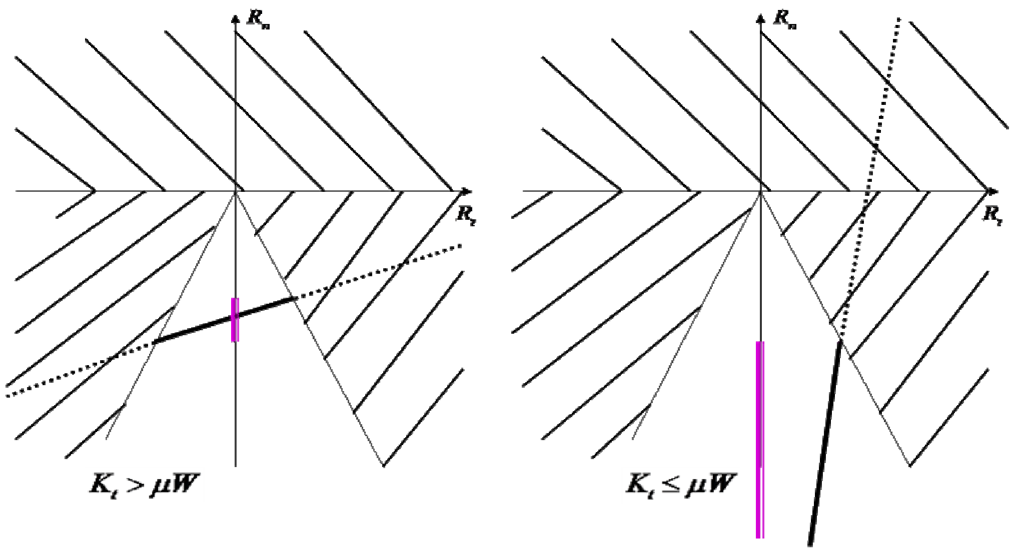

Figure 3: Examples of the set of stationnary solutions in the $\{R_t, R_n\}$ plane

Examples of the set of solutions that may occur are represented by the thick black line in figure 3, while the thick vertical line on the R_n axis is the corresponding set of normal reactions, the latter being useful in the following.

We first discussed the stability of these equilibria. Since the contact and friction laws rule out the possibility of using classical stability theorems, the first approach to stability consisted in computing directly the trajectories starting from initial data close to each of the equilibria and calculating if the trajectories diverge or not from the equilibria. But the limit of such a study soon became obvious. To wit, applying a small initial sliding velocity to a stationary solution strictly inside the Coulomb cone has no sense until the reactions are brought to one side of the cone, which may require an important force. This means in turn that any sufficiently small perturbation of the force of an equilibrium strictly inside the cone will not set the mass into motion. This phenomenon is of course due to the frictional contact and would not appear if the friction law had been regularized. A new notion of stability based on perturbations of the loading was then introduced in [9]. At this point it was interesting to study the response of the system to an oscillating loading.

4. QUALITATIVE DYNAMICS

The mass is submitted to a constant external force for which the set of equilibria is the one represented in figure 3 for $K_t > \mu W$ and an additional oscillating tangential force of rectangular wave shape is applied to the system. The choice of such a loading enables us to complete relatively complex analytical computations and thus obtain a clear vision of the response in the {period, amplitude} plane. Let ε be the amplitude of this loading and T its half period. As noticed above, the mass may remain at equilibrium. Using the equilibrium equation deduced from problem (1), it is easy to obtain that under such a loading, the line which represents the set of equilibria oscillates in the $\{R_t, R_n\}$ plane at the period of the perturbation.

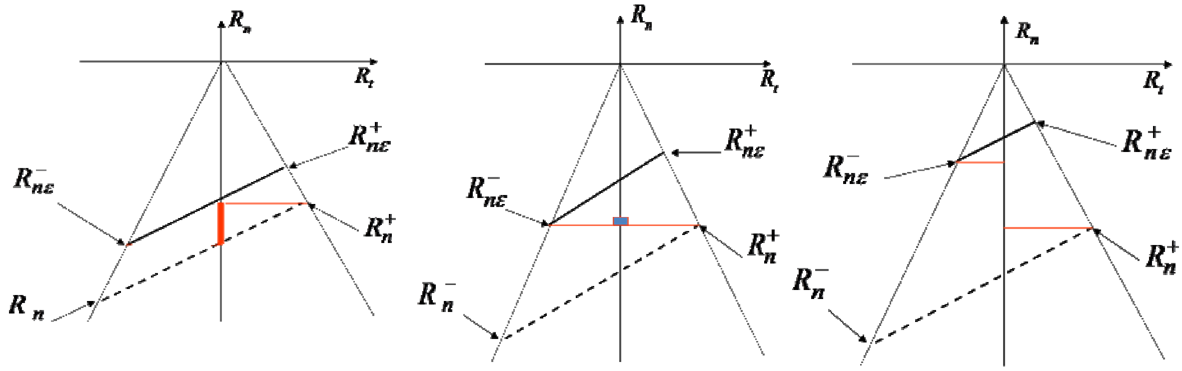

Figure 4: The effect of the amplitude of the loading

On figure 4 the following notations are used:

$$R_n^- = \frac{-A}{K_t - \mu W}, R_{n\varepsilon}^- = \frac{-A + \varepsilon W}{K_t - \mu W}, \quad R_n^+ = \frac{-A}{K_t + \mu W} \quad and \quad R_{n\varepsilon}^+ = \frac{-A + \varepsilon W}{K_t + \mu W}.$$

This is shown on figure 4 for three values of the amplitude where the set of equilibria oscillates from the dotted line to the thick continuous line. We represent by a thick line on the R_n axis the normal reactions which are common to the two sets of equilibria, i.e. which corresponds to an equilibrium for all time. For a sufficiently small amplitude ε (on the left) this line consists of a whole interval on the R_n axis, which means as noticed previously that the load will oscillate without setting the mass into motion. Increasing the amplitude, the line reduces to a single point (at the centre of figure 4) and increasing again the amplitude there no longer exists a common value of R_n (on the right) so that no equilibrium can be obtained under the corresponding oscillating force.

The global results, which correspond to a partition of the $\{T,\varepsilon\}$ plane, are the following:

- The first one is immediately deduced from the calculations which give figure 4. There exists a critical value $\varepsilon_0 = \frac{2\mu A}{K_t + \mu W}$ of the amplitude of the oscillation such that there exist infinitely many equilibrium states for any $\varepsilon < \varepsilon_0$. In this range, all trajectories issued from initial data out of equilibrium lead to an equilibrium in finite time. It is worth noticing that the boundary ε_0 of this lower range does not depend on the period of the excitation.
- The second one deals with the qualitative behavior on the horizontal line $\varepsilon = \varepsilon_0$. As represented on figure 4, there exists a single stationary solution for any T. Moreover, on this line, there exists a critical value T_0 of the period such that:
  - for $T < T_0$ all trajectories issued from initial data out of equilibrium lead to the unique equilibrium at infinity,
  - for $T > T_0$ in addition to the single equilibrium there also exist infinitely many periodic solutions.
- The third one deals with the range $\varepsilon > \varepsilon_0$ where stationary solutions no longer exist as can be seen on figure 4. We shall show that there exist different types of periodic solutions. At first we determine a range where the periodic solutions involve only sliding motions, without any loss of contact. An important difference between this range and the one where there exists infinitely many equilibria is that the upper boundary of this range depends on the period in a relativeley complicated way. Beyond this boundary, the periodic solutions will loose contact so that they will involve not only friction, but also impacts. The next subsections will describe these two ranges.

4.1 Sliding periodic solutions

All the explicit numerical values given in this section will be obtained for the following set of data: $m = 1,\ \mu = 0.5,\ K_t = 2,\ K_n = 3,\ W = 1,\ F_n = 2,\ F_t = 1.$

The line $\varepsilon = \varepsilon_0$ (=1.2) is the transition between the existence of infinitely many equilibrium states and the non existence of equilibria. On this line there exists a single equilibrium state and no non trivial periodic solution as long as the period T is strictly smaller than a critical value T_0, and when T is larger than T_0 the single equilibrium state coexists with infinitely many periodic solutions all contained inside a periodic solution of greatest amplitude. This is represented in the phase space of the tangential motion in figure 5 where the left part shows a trajectory which leads to the single equilibrium state at infinity, and the right part shows some periodic trajectories, all diffeomorphic to an ellipse around the unique equilibrium together with a non periodic trajectory which in this case tends to the periodic trajectory of greatest amplitude at infinity.

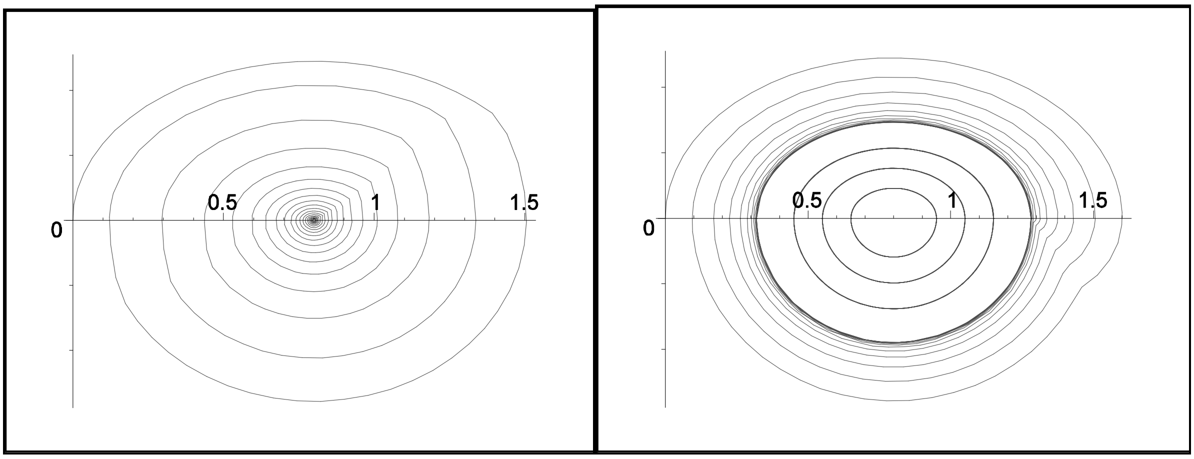

Figure 5: Trajectories for $\varepsilon = \varepsilon_0$ and for $T = 2$ (left) and $T = 3.2$ (right), (the numerical values used here lead to $\varepsilon_0 = 1.2$ and $T_0 = 2.56$).

This situation does not fit the classical dynamical systems theory. On the one hand at first glance the point $T = T_0$ on the $\varepsilon = \varepsilon_0$ axis could be seen as a codimension 2 Hopf bifurcation, but on the other hand two points are unusual:

- the first one is that, whether ε increases up to ε_0 on the $T = T_0$ axis or whether T increases up to T_0 on the $\varepsilon = \varepsilon_0$ axis, equilibrium states exist under a periodic loading which would not be the case for smooth dynamical systems,
- the second one is that there exists a maximal size periodic orbit such that any initial data inside the domain bounded by this orbit is on a periodic solution, which is classical, while initial data out of this domain leads to a non periodic solution, which is not classical.

4.1.1 When no equilibrium state exists.

Since the restoring force Ku simply involves a 2 x 2 matrix, each phase of the motion during which the sign of the velocity does not change is the solution of a linear ordinary differential equation. More precisely, the equation of motion in problem (1) reads :

$$\begin{cases} i) & m\,\ddot{u}_t + K_t u_t + W u_n = F_t + R_t\,, \\ ii) & m\,\ddot{u}_n + W u_t + K_n u_n = F_n + R_n\,. \end{cases} \tag{2}$$

According to the sliding direction, we have either $R_t = \mu\, R_n$ or $R_t = -\mu R_n$, so that we have:

$$either \begin{cases} u_n \equiv 0, \\ m\ddot{u}_t + (K_t - \mu W)u_t = F_t - \mu F_n, \end{cases} or \begin{cases} u_n \equiv 0, \\ m\ddot{u}_t + (K_t + \mu W)u_t = F_t + \mu F_n. \end{cases} \tag{3}$$

and thanks to the theoretical results recalled in section 3 the trajectory can be calculated piecewisely. Its velocity is continuous (not simply of Bounded Variation!) and there are jumps of the tangential reaction whenever the velocity changes sign.

4.1.2 Sliding periodic solutions

We choose a loading which consists of a constant part of components F_t, F_n and an oscillating part of period *2T* and amplitude ε. Let ω_α (respectively ω_β) be the frequency of the free oscillations with respect to the stiffness of the sliding to the right (respectively to the left). So that : $\omega_\alpha = \sqrt{\frac{K_t - \mu W}{m}}$ and $\omega_\beta = \sqrt{\frac{K_t + \mu W}{m}}$. Then calculating a trajectory over a period of the loading amounts to solving a system of the following form where m is set equal to one:

$$\begin{cases} \ddot{u}_1 + \omega_\alpha^2 u_1 = F_t - \mu F_n + \varepsilon, \quad t \in [0, \tilde{t}] \\ u_1(0) = u_0, \quad \dot{u}_1(0) = v_0, \\ \tilde{t} \text{ such that } \dot{u}_1(\tilde{t}) = 0, \\ \\ \ddot{u}_2 + \omega_\beta^2 u_2 = F_t + \mu F_n + \varepsilon, \quad t \in [\tilde{t}, T] \\ u_2(\tilde{t}) = u_1(\tilde{t}), \quad \dot{u}_2(\tilde{t}) = 0, \\ \\ \ddot{u}_3 + \omega_\beta^2 u_3 = F_t + \mu F_n, \quad t \in [T, \hat{t}] \\ u_3(T) = u_2(T), \quad \dot{u}_3(T) = \dot{u}_2(T) \\ \hat{t} \text{ such that } \dot{u}_3(\hat{t}) = 0, \\ \\ \ddot{u}_4 + \omega_\alpha^2 u_4 = F_t - \mu F_n, \quad t \in [\hat{t}, 2T] \\ u_4(\hat{t}) = u_3(\hat{t}), \quad \dot{u}_4(\hat{t}) = 0, \end{cases} \tag{4}$$

where $\tilde{t}, \hat{t}$ are unknowns corresponding to the changes of sign of the velocity, while T is given. Of course there is no reason in general for the mass to oscillate in phase with the phase changes of the oscillating force. A periodic solution shall exist, for example of the same period as that of the excitation, if the following system, where the unknowns are the initial data (u_0, v_0), has a solution

$$\begin{cases} u_4(2T) = u_0, \\ \dot{u}_4(2T) = v_0. \end{cases} \tag{5}$$

In this abstract form, equation (5) seems simple, but it is in fact a relatively intricate algebraic system since $u_4(2T)$ and $\dot{u}_4(2T)$ depend on the initial data through all the phases of the sliding motion given by equation (4). Writing that a trajectory starting from some given initial data comes back to this initial data after some period gives consequently an equation for the initial data which is complicated but algebraic, as soon as the period is given. So that finding periodic solutions requires an investigation with respect to the period. These calculations can be performed analytically in some cases when the period is sufficiently large for the mass to stay at rest before each change of the excitation, but the results given in general will be calculated using the software Maple with the numerical values of the parameters given above. Two examples are shown in figure 6.

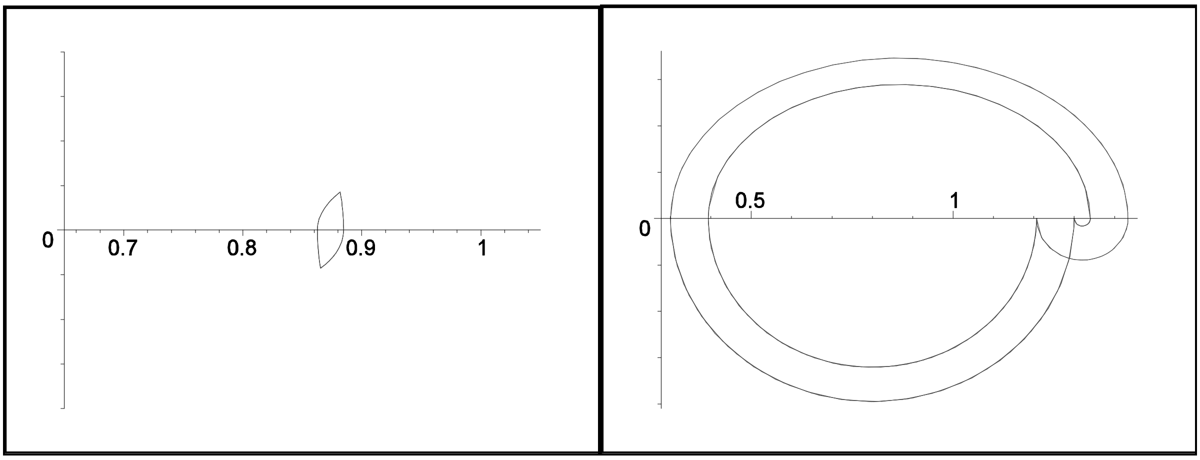

Figure 6: Periodic solutions represented in the phase plane for $\varepsilon > \varepsilon_0$ and two different values of the period of the excitation ($T = 0.5$ and $T = 7$).

Two points are worth noticing:

- In the case of small T (figure 6 left), equation (5) has a single solution for each pair of data (T,ε). So that there exists a single periodic trajectory. It has the period of the excitation but involves a phase difference with respect to the excitation which increases as the period of the excitation decreases. Represented in the phase space, this trajectory is homeomorphic to an ellipse, with vertexes corresponding to jumps of the acceleration at the changes of the external load, and is of smaller and smaller amplitude either when ε increases or when T decreases.
- In the case of large T (figure 6 right) there exist infinitely many periodic solutions of period twice the period of the excitation. This result follows from the fact that in the phase space $\{u_t, \dot{u}_t\}$ initial data in a whole interval on the u_t axis are compatible with the periodicity condition (5). So that for each given pair (T,ε) of data in a strip of the type $]\varepsilon_0, \tilde{\varepsilon}[\times \left]\frac{2\pi}{\omega_\alpha} + \frac{2\pi}{\omega_\beta}, +\infty\right[$ there exist infinitely many periodic solutions of period *4T*, and it has been obtained in [8] that everywhere in this strip these solutions coexist with a single periodic solution of period *2T*.

4.1.3 The loss of contact boundary

We have obtained a periodic response of the system for any value of the period of the excitation. The important point is now to determine exactly for what amplitudes of the excitation we can be sure that the mass stays in contact. We shall first obtain that there actually exists a limit amplitude beyond which all the periodic solutions loose contact at least during some part of the period, and secondly that this limit depends strongly on the value of the period. The main steps are the following

- The first step consists in observing on the one hand that a necessary condition for a particle to loose contact is that its reaction crosses the vertex of the cone that is the normal component of its reaction attains zero, and on the other hand that having a sliding trajectory solution to system (4-5), equation (2-*ii*) with $u_n \equiv 0$ implies that during a sliding motion $R_n = 0$ is reached for $u_t = \frac{F_n}{W}$.

- In the second step we use the fact that, due to unilateral contact, R_n must be negative so that $R_n = 0$ is its maximal value. Since u_t and R_n are associated through the relation $R_n = Wu_t - F_n$ during a sliding motion, the maximal value of the normal reaction, $R_{n\,max}$, corresponds to the maximal value of u_t.
- We thus choose a pair (T, ε), solve problem (4) and satisfy the periodicity condition (5), then calculate the maximal value of u_t during the sliding periodic motion and deduce the corresponding $R_{n\,max}$. All this procedure can be considered as a map which associates the maximal value of R_n with any pair (T, ε). Let $\varphi(T, \varepsilon)$ be this map.
- Finally, the loss of contact occurs when

$$\varphi(T, \varepsilon) = 0 \qquad (6)$$

Equation (6) defines implicitely the boundary beyond which purely sliding periodic solutions no longer exist since the mass begins to loose contact and jumps. Let $\varepsilon = \varepsilon(T)$ be the implicit solution of equation (6). Again we used the sofware Maple to calculate this boundary piecewisely according to the different types of periodic solutions.

The next subsection deals with what happens beyond this boundary and how to calculate periodic solutions in this zone.

4.2 Periodic solutions involving loss of contact

When the trajectories loose contact the calculations become more intricate. At first because, as proved by elementary examples (see e.g. [1]), a trajectory may involve impacts which accumulate at some time even under extremely smooth or constant loading. Secondly even in smoother cases where no accumulation of impacts occur, rewritting problem (4) in order to deal both with the tangential and the normal component will not give a closed problem since the initial tangential velocity of the phase of motion which just follows an impact is not given by system (1).

We first briefly describe a numerical scheme which is efficient even when there exists an infinite number of impacts in some time intervals, then we show how to calculate the velocity after an impact. After that an example of a trajectory involving loss of contact is given where we see that the tangential velocity is discontinuous at the impact times and can even jump from a strictly negative value (or strictly positive one) to zero.

4.2.1 The time stepping procedure

Problem (1) is set on the time interval $[0,T]$. Let us divide this interval into time steps (t_i, t_{i+1}) of a given size h. Since the equation of motion in problem (1) must be understood in the sense of measures, it can be integrated over any time interval. Let us choose an interval of the form $]t_i, t_{i+1}]$. From the definition of a measure, the primitive of the acceleration $\ddot{u}$ is a function of bounded variation, which means that the velocity $\dot{u}$ has everywhere on $[0,T]$ a right and a left limit. Let $\dot{u}^+(\tau)$ be its right limit at any point τ in $[0,T]$. We get:

$$m(\dot{u}^+(t_{i+1}) - \dot{u}^+(t_i)) = \int_{t_i}^{t_{i+1}} F\,ds - \int_{t_i}^{t_{i+1}} Ku\,ds + \int_{]t_i,t_{i+1}]} R \qquad (7)$$

Following Moreau and Jean (see [7] and [5]), let us introduce as new unknowns a tangential (respectively normal) percussion $P_t^{i+1} := \int_{]t_i,t_{i+1}]} R_t$ (respectively $P_n^{i+1} = \int_{]t_i,t_{i+1}]} R_n$). Whatever the choice of the integration scheme choosen for $\int_{t_i}^{t_{i+1}} Ku\, ds$, system (7) can be rewritten in the following way:

$$\begin{cases} (i) & \dot{u}_n^+(t_{i+1}) = E_n + \pi_n P_n^{i+1}, \\ (ii) & \dot{u}_t^+(t_{i+1}) = E_t + \pi_t P_t^{i+1}, \end{cases} \tag{8}$$

where π_n and π_t are constants depending on the mass, the stiffness parameters and the integration scheme, and where E_t and E_n include everything which is known at time t_i. System (8) has four unknowns but only two equations so that two relations must be added, one between $\dot{u}_n^+(t_{i+1})$ and P_n^{i+1}, another one between $\dot{u}_t^+(t_{i+1})$ and P_t^{i+1}. Different relations can be used. For example the relation between $\dot{u}_t^+(t_{i+1})$ and P_t^{i+1} can be obtained by imposing that the point $\left(\dot{u}_t^+(t_{i+1}),\ P_t^{i+1}\right)$ belongs to the graph of the Coulomb law so that $\dot{u}_t^+(t_{i+1})$ and P_t^{i+1} are then obtained simply by calculating the intersection of the straight line (8-*ii*) and this graph, see [5].

4.2.2 How to deal with impact times

We first observe that the initial problem (1) involves an impact law which directly gives the normal component of the velocity after an impact if the velocity before the impact is known (we recall that the functional framework implies that both limits exist at any time) as soon as the physical restitution coefficient *e* is given. This means that only the tangential component of the velocity after an impact remains to be calculated. Let us come back to equation (7) but reducing the time interval (t_i, t_{i+1}) to the zero Lebesgue measure interval consisting of a single impact time. Let τ be such a time. Of course the integral of the functions *Ku* and *F* over the interval $\{\tau\}$ vanish and system (7) is simplified into:

$$\begin{cases} (i) & m\,\{\dot{u}_n^+(\tau) - \dot{u}_n^-(\tau)\} = R_n(\tau) \\ (ii) & m\,\{\dot{u}_t^+(\tau) - \dot{u}_t^-(\tau)\} = R_t(\tau). \end{cases} \tag{9}$$

From equation (9-*ii*) we deduce the following linear relation :

$$\dot{u}_t^+(\tau) = \frac{1}{m} R_t(\tau) + \dot{u}_t^-(\tau)$$

in the $\{\dot{u}_t^+, R_t\}$ plane, which we intersect with the graph of the friction law where $\mu R_n(\tau)$ is obtained by equation (9-*i*) and the impact law of problem (1).

4.2.3 Periodic solutions involving impacts

Figure 7 represents a periodic solution involving shocks where the excitation has the same period as for the periodic solution presented on figure 6 right but has an amplitude strictly larger than the limit $\varepsilon(T)$ where contact is lost. The normal component of the displacement does not remain identically equal to zero as shown on figure 7 right (where we also see that it satisfies the unilateral contact condition) but the main point worth noticing is seen on Figure 7 left. In the same way as for $\varepsilon < \varepsilon(T)$ for the same value of *T*, the period of the response is twice the one of the excitation.

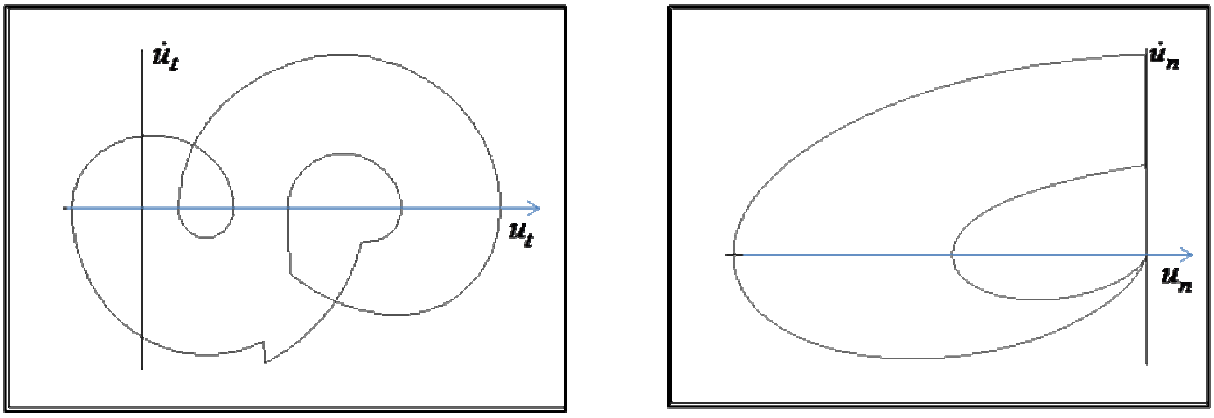

Figure 7: Tangential and normal phase plane for a periodic solution which looses contact

But now we see that the loss of contact, i.e. the time intervals where u_n is strictly negative, ends by an impact associated with a jump of $\dot{u}_t$. Such impacts occur twice during the period (we can see on figure 7 right that the normal velocity is not zero at these impact times) but we also see on figure 7 left that at one of the impacts the tangential velocity just reduces its amplitude, whereas at the other impact the velocity jumps from a stricly negative value to zero. All the results of this section can be summarized in the following figure.

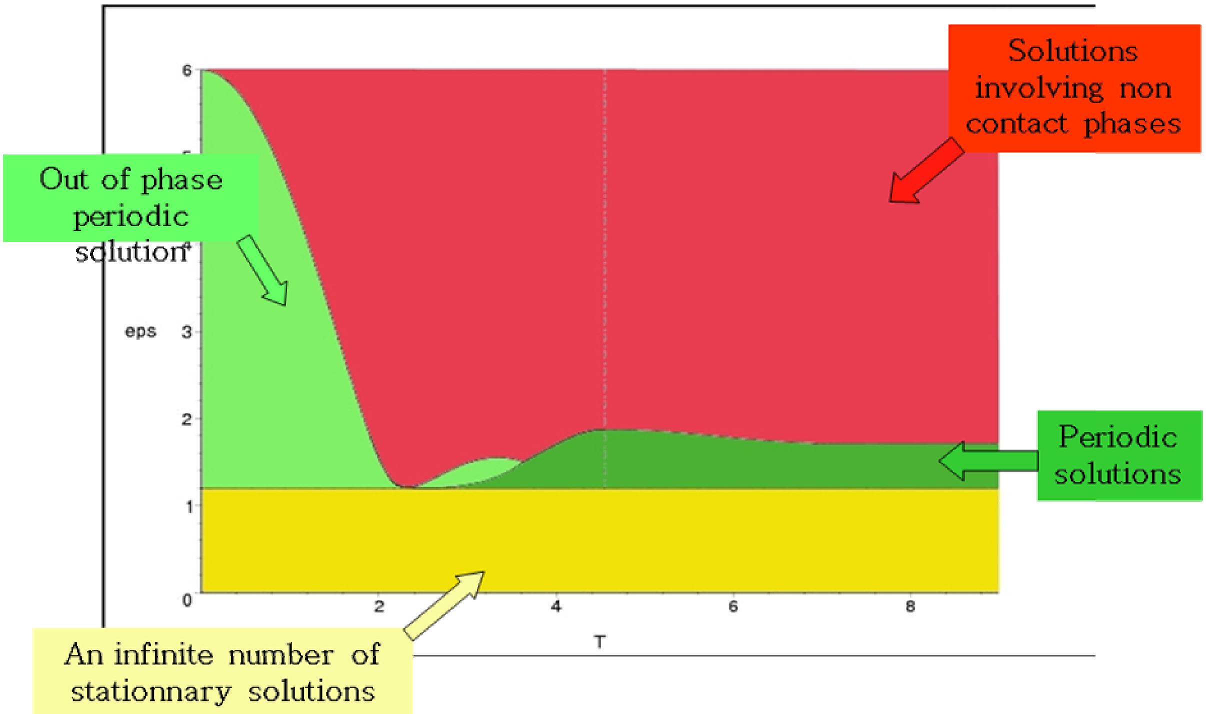

Figure 8: Response to the oscillating loading in the $\{T, \varepsilon\}$ plane

5. CONCLUSION

As a first concluding remark on the qualitative dynamics, let us comment on the global picture of figure 8. We can in particular see that for small periods of the excitation the boundary $\varepsilon(T)$ increases strongly and it can easily be calculated that the boundary reaches values which are more than three times the limit of static equilibrium in contact. This means that we can apply an excitation of very large amplitude without loosing contact under the condition that the frequency of this excitation is high enough. Conversely, another point to be noticed is that the boundary $\varepsilon(T)$ decreases to a very low value, close to the limit of equilibrium, for intermediate values of *T* (close to *T=2* on figure 8). It has been shown in [8] that around this minimum, which might be interpreted as a resonnance, the dynamics involve several very thin layers with different kinds of periodic sliding solutions.
An important point studied in the present work was the occurence of the loss of contact and the existence of periodic trajectories partly out of contact. In this range, trajectories not only involve friction but also impacts and this study required a specific analysis to establish how the right limit of the velocity of the solution at impact times can be obtained from the left limit in a manner compatible with the unilateral contact law and the friction law.
This study suggests a certain number of further research subjects. First of all, while the stability analysis in the sliding case is at its beginning and is completely open in the case with impacts, a thorough stability analysis is needed to attain a full understanding of the dynamics. Another important question is whether there exists ranges of the parameters where no periodic solutions exist and if so determining eventual transisitions to chaos. The present study involves only nonsmooth nonlinearities, due to contact and friction, whereas smooth nonlinearities, such as large strains, have been largely studied through classical nonlinear dynamics. We are currently working on the coupling of these two nonlinearities, coupling which appears in many physical systems.

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