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Length scale effect analysis on vibration behavior of single walled Carbon NanoTubes with arbitrary boundary conditions

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Résumé

Dans cet article, le comportement vibratoire des nanotubes en carbone (CNT) a été modélisé et numériquement évalué en se basant sur la théorie de l'élasticité non locale et le modèle de poutre de Timoshenko. Des conditions aux limites généralisées ont été considérées afin de prendre en compte un large éventail de conditions aux limites. Les basses et les hautes fréquences ainsi que les modes propres associés peuvent être obtenus par résolution numérique de l'équation transcendante présentée. Les effets de taille sur les fréquences et modes propres ainsi que l'analyse paramétrique en fonction des paramètres géométriques et matériels des nanotubes peuvent être facilement effectués. Pour les nanotubes encastrées-libres, des valeurs critiques du paramètre d'échelle au delà desquelles des instabilités de type flutter peuvent être obtenues ont été données. La coalescence des modes propres paires correspondants à ces valeurs critiques est adressée. Il a été montré que ce type d'instabilité peut aussi être rencontré pour des conditions aux bords généralisées.

Abstract

In this paper the small length scale effects on the vibration behaviors of single walled Carbon Nano Tubes (CNT) are modeled and numerically evaluated based on the nonlocal elasticity theory and the Timoshenko beam model. Generalized boundary conditions are considered in order to take into account a more realistic and a wide range of boundary conditions. The lower as well as higher natural frequencies and associated eigenmodes can be obtained by numerically solving the presented generalized transcendental nonlinear algebraic equation. The effect of the used translational and the rotational spring constants on the vibration frequencies and mode shapes of the CNT are addressed. It is demonstrated that the small scale effect can lead to an unstable behaviors of flutter type for cantilever CNT. The coalescence of pair eigenmodes is addressed at critical length scales and these critical values are decreased by increasing the mode number.

1. INTRODUCTION

Carbon NanoTubes (CNTs) have become one of the most promising material and appear to possess extraordinary physical properties. Many applications of CNTs have been reported, such as in atomic force microscopes (AFMs), sensors, actuators, resonators, nano oscillators and field emission devices. To realize the potential benefits of CNTs a fundamental understanding of nano-structured material is required in order to develop reliable constitutive models for various design purposes. The modeling for CNT is classified into two main categories. The first one is mainly based on atomistic and molecular dynamics simulations. The second one is the continuum modeling, including local and nonlocal beam or shell theories. Successful works have been concluded with continuum modeling, such as nonlocal elasticity and mechanical property investigation of CNTs [1-5]. These papers indicate that the small length scales would have significant influences and the nonlocal continuum model can effectively capture these influences in the study of nanostructures.

The Euler-Bernoulli and Timoshenko beams models combined with nonlocal elasticity theory are often used for the analysis of CNT beam like flexural motions. But, for CNT in which the radial and/or circumferential displacements have more or less affects, the nonlocal elastic shell theory is more adapted. The wave propagation of single and multi walled CNT has been investigated by Wang and Varadan [6] and Mitra and Gopalakrishnan [7] based on the nonlocal shell theory. An assessment of the continuum beam and shell models in the buckling prediction of CNT is provided by Zhang et al. [8] on the basis of molecular dynamic simulation results as well as on some published results. It was demonstrated that the accuracy of the beam and shell models depends on the CNT's aspect ratios (Length/diameter, L/d) and diameters. For large aspect ratio (L/d >10) the beam model is largely enough. But, for short CNT with large diameter the nonlocal shell models are more adapted.

Vibrations of CNTs are of considerable importance in a number of mechanical devices and occur during certain manufacturing processes of nanocomposites. So, there are considerable motivation for studying vibration characteristics of CNTs. Gibson et al [1], Benzair et al [9] and Civalek et al [10], among others, discussed the free vibration and bending analyses of CNTs based on the nonlocal continuum model. Azrar et al [11] developed higher order free vibration characteristics of single walled carbon nanoTubes based on the nonlocal Timoshenko and Bernoulli beam models. The mathematical developments and numerical predictions based on these two models have been deeply analyzed with respect to aspect ratios and small length scale. Adali [12] and Kucuk et al. [13] elaborated variational principles of CNTs based on Bernoulli and Timoshenko models. Pin Lu et al. [14] studied the dynamic properties of flexural beam using the nonlocal elasticity model. Yoon et al. [15] developed the vibration of double walled short carbon nanotubes of small aspect ratios modeled by the Timoshenko model and neglecting the nonlocal effect and using the classical boundary conditions. The vibration analyses of classical beams under generalized boundary conditions are addressed by Li [16, 17] using the Bernoulli model and by Arboleda-Monsalve et al. [18] using the Timoshenko model. In this paper the vibration characteristics of CNT under arbitrary boundary conditions based on the Timoshenko model and the nonlocal elasticity theory are investigated. The main target of the present work is to investigate the length small scale and the generalized boundary conditions effects on the eigenfrequencies and eigenmodes as well as on the instability of these nanostructures.

2. MATHEMATICAL FORMULATION

Let us consider a slender single walled carbon nanotube of length L, diameter d and thickness h under arbitrary boundary conditions described by translational and rotational springs at both ends. This allows describing the more realistic boundary conditions and covering the classical boundary conditions by simply specifying the spring constants. The nonlocal elasticity theory combined with the Timoshenko beam model is adopted.

2.1 Constitutive and governing dynamic equations

Based on the nonlocal elasticity theory, developed by Eringen [2,3], the constitutive equation for a linear homogenous nonlocal elastic body is given by the following integral equation:

$$\sigma_{ij}(x) = \int_V \alpha(|x - x'|, \tau)\, C_{ijkl}\varepsilon_{kl}(x')\, dV(x'), \tag{1}$$

$$\varepsilon_{kl} = (u_{k,l} + u_{l,k})\,/\,2 \tag{2}$$

in which $\alpha(|x - x'|, \tau)$ is the nonlocal kernel function, which incorporates the nonlocal effect at the reference point x produced by local strain point at the source x' into the constitutive equations $\tau = e_0 a / L$ in which a and L are internal and external characteristic lengths and e_0 is a constant appropriate to each material. C_{ijkl} is the elastic modulus tensor, σ_{ij} *and* ε_{ij} are the stress and the strain tensors. The nonlocal kernel function α depends on the internal and external characteristics lengths. Various approximate models of nonlocal elasticity can be obtained by specifying the kernel function α. This leads to a constitutive equation in an integral from. For practical reasons, the following differential constitutive equations for one dimensional case with Timoshenko hypothesis and small deflection are adopted [2,3].

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \tag{3}$$

$$\sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G\gamma_{xz}, \tag{4}$$

where $\sigma_{xx}, \sigma_{xz}, \varepsilon_{xx}$, γ_{xz}, E and G are the nonlocal stress tensors, axial and transverse shear strains, Young modulus and shear modulus respectively. The displacement components along the axial x and transverse z directions are denoted by U and W respectively:

$$U(x,z,t) = u(x,t) - z\psi(x,t), \quad W(x,z,t) = w(x,t) \tag{5}$$

where ψ denotes the rotation of the cross section, u and w are the middle plane (z=0) components and t is time. The axial strains are:

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + z\frac{\partial \psi}{\partial x}, \; \varepsilon_{xx}^0 = \frac{\partial u}{\partial x}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \psi\,, \tag{6}$$

Based on these equations, the nonlocal resultant axial force N, moment M and shear force Q are obtained by the following differential equations.

$$N - (e_0 a)^2 \frac{\partial^2 N}{\partial x^2} = EA\varepsilon_{xx}^0, \tag{7}$$

$$M = (e_0 a)^2 \frac{\partial^2 M}{\partial x^2} + EI\frac{\partial \psi}{\partial x}, \tag{8}$$

$$Q = (e_0 a)^2 \frac{\partial^2 Q}{\partial x^2} + ksAG\left(\frac{\partial w}{\partial x} + \psi\right), \tag{9}$$

in which $$N = \int_A \sigma_{xx}\, dA, \quad M = \int_A z\,\sigma_{xx}\, dA, \qquad Q = ks\int_A \sigma_{xz}\, dA \tag{10}$$

where ks is the factor of shear depending on the shape of the cross section A. The equations of motion for the transversely vibration of CNT is given by:

$$\frac{\partial V^T(x,t)}{\partial x} + q(x,t) = m_0 \frac{\partial^2 w(x,t)}{\partial t^2}, \tag{11}$$

$$\frac{\partial M(x,t)}{\partial x} - V^T(x,t) = m_2 \frac{\partial^2 \psi(x,t)}{\partial t^2}, \tag{12}$$

where $q(x,t)$ is the transverse excitation force per unit length. For a constant cross section, the mass inertia m_0 *and* m_2 are defined by: $m_0 = \int_A \rho dA = \rho A$; $m_2 = \int_A \rho z^2\, dA = \rho A \frac{h^2}{12}$, where ρ, h, *and* A are the mass density of the material, the tube thickness and the cross section area respectively. Using the previous equations, the following moment and shear forces are obtained.

$$M = EI\frac{\partial \psi}{\partial x} + (e_0 a)^2 \left[m_2 \frac{\partial^3 \psi}{\partial x \partial t^2} + m_0 \frac{\partial^2 w}{\partial t^2} - q \right], \tag{13}$$

$$V^T = ksAG\left(\psi + \frac{\partial w}{\partial x}\right) + (e_0 a)^2 \left[m_0 \frac{\partial^3 w}{\partial x \partial t^2} - \frac{\partial q}{\partial x} \right], \tag{14}$$

where I is the second moment depending on the considered shape of the cross section.
Substituting Eqs. (13) and (14) into (11) and (12) one obtains the partial differential system governing the dynamic behavior of undamped CNTs.

$$\begin{cases} EI\dfrac{\partial^2 \psi}{\partial x^2} - ksAG\left(\dfrac{\partial w}{\partial x} + \psi\right) = m_2 \dfrac{\partial^2}{\partial t^2}\left[\psi - (e_0 a)^2 \dfrac{\partial^2 \psi}{\partial x^2}\right], \\ ksAG\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial x} + \psi\right) + q - (e_0 a)^2 \dfrac{\partial^2 q}{\partial x^2} = m_0 \dfrac{\partial^2}{\partial t^2}\left[w - (e_0 a)^2 \dfrac{\partial^2 w}{\partial x^2}\right], \end{cases} \tag{15}$$

Recall that the classical local Timoshenko beam model is recovered when the parameter e_0 is set to zero. The vibration analysis of the CNT will be based on these equations.

3. FREE VIBRATING MODELING

For linear free vibrations, let us assume that.

$$w(x,t) = W(x)e^{i\omega t}, \qquad \psi(x,t) = \Psi(x)e^{i\omega t} \; and \qquad q(x,t) = 0, \tag{16}$$

where ω is the natural vibration frequency parameter. The substitution of Eqs. (16) into (15) leads to the following coupled differential equations.

$$\begin{cases} EI\dfrac{d^2\Psi}{dx^2} - ksAG\left(\dfrac{dW}{dx} + \Psi\right) + m_2\omega^2\left[\Psi - (e_0 a)^2 \dfrac{d^2\Psi}{dx^2}\right] = 0, \\ ksAG\dfrac{d}{dx}\left(\dfrac{dW}{dx} + \Psi\right) + m_0\omega^2\left[W - (e_0 a)^2 \dfrac{d^2 W}{dx^2}\right] = 0, \end{cases} \tag{17}$$

The coupled differential system (17) can be reduced to the following uncoupled fourth order differential equations [11].

$$p\frac{d^4W}{dx^4}+q\frac{d^2W}{dx^2}-rW=0 \qquad or \qquad p\frac{d^4\Psi}{dx^4}+q\frac{d^2\Psi}{dx^2}-r\Psi=0 \tag{18}$$

where $$p=\left(EI-(e_0a)^2m_2\omega^2\right)\left(1-\frac{(e_0a)^2m_0\omega^2}{ksAG}\right); \tag{19}$$

$$q=m_0\omega^2\left(\Omega_0+(e_0a)^2\right)+m_2\omega^2\left(1-2\frac{(e_0a)^2m_0\omega^2}{ksAG}\right);\ r=m_0\omega^2\left(1-\frac{m_2\omega^2}{ksAG}\right); \tag{20}$$

Based on the characteristic equation of (18), the following simplified frequency equation is obtained when the rotary mass inertia m_2 is neglected [11].

$$\omega_0=\alpha^2\left[\frac{EI}{m_0}\left(\frac{1}{(1+\Omega_0\alpha^2)\ (1+(e_0a)^2\alpha^2)}\right)\right]^{1/2}, \tag{21}$$

where $$\alpha^2=\frac{q+\sqrt{q^2+4pr}}{2p}, \qquad if \quad pr>0, \qquad \Omega_0=\frac{EI}{ksAG}, \tag{22}$$

$$\beta^2=\frac{\alpha^2}{1+\left(\Omega_0+(e_0a)^2\right)\alpha^2)}, \tag{23}$$

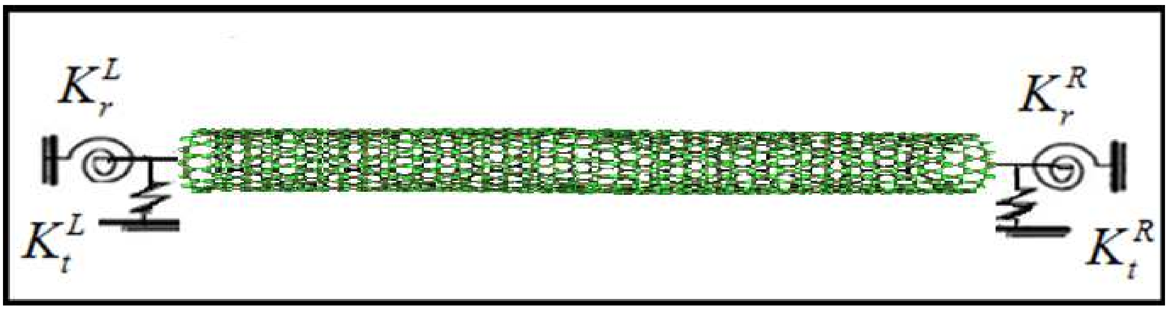
Note that α *and* β depend nonlinearly on the frequency ω.The associated eigenmodes deflection, moment and shear force are given by:

$$W(x)=A_1\sin(\alpha x)+A_2\cos(\alpha x)+A_3\sinh(\beta x)+A_4\cosh\ (\beta x) \tag{24}$$

$$M^T(x)=-EI\left(Q^4(\Omega_0+(e_0a)^2)W(x)+(1-Q^4\Omega_0(e_0a)^2)\frac{d^2W(x)}{dx^2}\right) \tag{25}$$

$$V^T(x)=-EI\left(Q^4(\Omega_0+(e_0a)^2)\frac{dW(x)}{dx}+(1-Q^4\Omega_0(e_0a)^2)\frac{d^3W(x)}{dx^3}\right) \tag{26}$$

where the unknown frequency parameter $Q=\sqrt{\omega_0\sqrt{\rho A/EI}}$ and the arbitrary constants A_i are determined by the considered boundary conditions.



For more realistic boundary conditions, the following generalized ones are considered:

$$M^T(0) - K_r^L \frac{dW(0)}{dx} = 0 \tag{27}$$

$$V^T(0) + K_t^L\, W(0) = 0 \tag{28}$$

$$M^T(L) + K_r^R \frac{d\,w(L,t)}{dx} = 0 \tag{29}$$

$$V^T(L,t) - K_t^R\, w(L,t) = 0 \tag{30}$$

where K_r^L, K_t^L *and* K_r^R, K_t^R are the translational and the rotational spring constants at the left and right ends of the CNT at x=0 and x=L (see figure 1). Equations (27-30) represent a set of generalized boundary conditions. The classical boundary conditions can be simply obtained as special cases when the stiffness's of the springs take some extreme values such as zero and infinity. For example, the clamped-clamped boundary condition can be easily obtained by assuming that at each end the translational and rotational spring constants are extremely large. Using equations (24-26) and the boundary conditions (27-30), one obtains the following algebraic system.

$$\begin{bmatrix} U_1\alpha & K_t^L & U_2\beta & K_t^L \\ -K_r^L\alpha & U_1 & -K_r^L\beta & U_2 \\ \left[U_1\alpha\cos(\alpha) - K_t^R\sin(\alpha)\right] & \left[-K_t^R\cos(\alpha) - U_1\alpha\sin(\alpha)\right] & \left[U_2\beta\cosh(\beta) - K_t^R\sinh(\beta)\right] & \left[U_2\beta\sinh(\beta) - K_t^R\cosh(\beta)\right] \\ \left[U_1\sin(\alpha) + K_r^R\alpha\cos(\alpha)\right] & \left[U_1\cos(\alpha) - K_r^R\alpha\sin(\alpha)\right] & \left[U_2\sinh(\beta) + K_r^R\beta\cosh(\beta)\right] & \left[U_2\cosh(\beta) + K_r^R\beta\sinh(\beta)\right] \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which U_1, U_2 are given by:

$$U_1 = -\alpha^2(1-(e_0a)^2Q^4\Omega_0)+(\Omega_0+(e_0a)^2)Q^4;$$

$$U_2 = \beta^2(1-(e_0a)^2Q^4\Omega_0)+(\Omega_0+(e_0a)^2)Q^4;$$

After some mathematical developments the following nonlinear algebraic transcendental equation is obtained:

$$F(\omega_n)=A+B\cos(\alpha_n)\cosh(\beta_n)+C\sin(\alpha_n)\sinh(\beta_n)+D\cos(\alpha_n)\sinh(\beta_n)+E\sin(\alpha_n)\cosh(\beta_n)=0 \tag{31}$$

where: $A=\alpha\beta((U_1^2+K_r^L K_t^L)(U_2^2+K_r^R K_t^R)+(U_2^2+K_r^L K_t^L)(U_1^2+K_r^R K_t^R));$

$$B=2\alpha\beta U_1U_2((K_r^R K_t^L - K_r^R K_t^R + K_t^R K_r^L - K_r^L K_t^L) - U_1U_2)$$
$$-\alpha\beta(K_r^R K_t^L + K_t^R K_r^L)(U_1^2+U_2^2) + K_r^R K_t^R K_t^L\alpha(U_1 - U_2 - 2K_r^L\beta);$$

$$C=(K_r^R K_r^L(\alpha\beta)^2 - K_t^R K_t^L)(U_1-U_2)^2 - (U_1U_2(U_1U_2+(K_r^R K_t^R + K_r^L K_t^L))$$
$$+K_r^R K_t^R K_r^L K_t^L)(\alpha^2-\beta^2);$$

$$D=((K_t^R - K_t^L)(U_1U_2 + K_r^R K_r^L\beta^2) - ((K_r^R - K_r^L)U_1U_2\beta^2 - K_t^R K_r^L K_t^L))\alpha(U_2-U_1);$$

$$E = (K_t^R - K_t^L)(U_1 U_4 (U_1 - U_2) + K_r^R K_r^L \alpha^2 \beta (U_2 - U_1))$$
$$+ (K_r^R - K_r^L)(U_1 U_2 \alpha^2 \beta (U_1 - U_2) - (U_1 - U_2) K_t^R K_t^L \beta);$$

This formulation allows one to get the natural frequencies with respect to the considered physical and material parameters. The numerical solutions of this equation are obtained by the Newton-Raphson algorithm. It is to be noted that when the small scale parameter $e_0a=0$, the characteristic equation is reduced to its classical from with generalized boundary conditions.
The resulting mode deflection and rotation shapes are given by:

$$W(x) = G_1 \sin(\alpha x) + G_2 \cos(\alpha x) + G_3 \sinh(\beta x) + \cosh(\beta x); \quad (32)$$
$$\psi(x) = S_{11}(G_1 \cos(\alpha x) - G_2 \sin(\alpha x)) - S_{22}(G_3 \cosh(\beta x) - \sinh(\beta x)); \quad (33)$$

where:

$$G_1 = \Big[\sinh(\beta)(U_1(U_2^2\beta^2 + K_t^L K_t^R) - K_t^L K_t^R U_2 + K_r^L K_t^L U_2 \beta^2) - \cosh(\beta)(U_1 \beta (K_t^L + K_t^R) U_2$$
$$- K_t^L \beta U_2^2 + K_r^L K_t^L K_t^R \beta) + K_t^R \beta \cos(\alpha)(U_2^2 + K_r^L K_t^L) + U_1 \alpha \beta \sin(a)(U_2^2 + K_r^L K_t^L)\Big]$$
$$/\Big[\beta \sin(\alpha)(K_r^L U2 U_1 \alpha^2 - K_r^L U_1^2 \alpha^2 + K_t^R U_2 U_1 + K_r^L K_t^L K_t^R) - \alpha \beta \cos(\alpha)(U_1^2 U_2 + K_r^L K_t^L U_1$$
$$+ K_r^L K_t^R U_1 - K_r^L K_t^R U_2) - K_t^R \alpha \sinh(\beta)(U_1^2 + K_r^L K_t^L) + U_2 \alpha \beta \cosh(\beta)(U_1^2 + K_r^L K_t^L)\Big];$$

$$G_2 = -\Big[\cosh(\beta)(\alpha(K_r^L K_t^L U_2 \beta + K_r^L K_t^R U_2 \beta) + U_1 \alpha (U_2^2 \beta - K_r^L K_t^R \beta))$$
$$- \sinh(\beta)(\alpha (K_r^L U_2^2 \beta^2 + K_r^L K_t^L K_t^R) + U_1 \alpha (K_t^R U2 - K_r^L U_2 \beta^2)) + K_t^R \beta \sin(\alpha)(U_2^2 + K_r^L K_t^L)$$
$$- U_1 a \beta \cos(a)(U_2^2 + K_r^L K_t^L)\Big]/\Big[\beta \sin(\alpha)(K_r^L U_2 U_1 \alpha^2 - K_r^L U_1^2 \alpha^2 + K_t^R U_2 U_1 + K_r^L K_t^L K_t^R)$$
$$- \alpha \beta \cos(\alpha)(U_1^2 U_2 + K_r^L K_t^L U_1 + K_r^L K_t^R U_1 - K_r^L K_t^R U_2) - K_t^R \alpha \sinh(\beta)(U_1^2 + K_r^L K_t^L)$$
$$+ U_2 \alpha \beta \cosh(\beta)(U_1^2 + K_r^L K_t^L)\Big];$$

$$G_3 = -(K_t^R \cosh(\beta) - ((U_1 U_2 + K_r^L K_t^L)(K_t^R \cos(\alpha) - U_1 \alpha \sin(\alpha)))/(U_1^2 + K_r^L K_t^L)$$
$$+ U_2 \beta \sinh(\beta) - (K_t^L (U_1 - U_2)(K_t^R \sin(\alpha) + U_1 \alpha \cos(\alpha)))/(\alpha (U_1^2 + K_r^L K_t^L)))/(K_t^R \sinh(\beta)$$
$$+ (K_r^L \beta (U_1 - U_2)(K_t^R \cos(\alpha) - U_1 \alpha \sin(\alpha)))/(U_1^2 + K_r^L K_t^L) + U_2 \beta \cosh(\beta)$$
$$- ((K_r^L K_t^L \beta + U_1 U_2 \beta)(K_t^R \sin(\alpha) + U_1 \alpha \cos(\alpha)))/(\alpha (U_1^2 + K_r^L K_t^L)));$$

$$S_{11} = \frac{-\alpha}{1 + \Omega_0 \alpha^2} \quad and \quad S_{22} = \beta \frac{\left(\Omega_0 + (e_0 a)^2\right)\alpha^2 + 1}{1 + (e_0 a)^2 \alpha^2}$$

Equations (31-33) give the mathematical modeling of the free vibration of CNT under generalized boundary conditions. The small as well as the higher order natural frequencies and associated eigenmodes can be obtained by numerically solving the considered general transcendental equation using the Newton Raphson algorithm. The small length scale as well as the generalized boundary conditions effects on the eigenfrequencies and eigenmodes can be analyzed.

4. NUMERICAL RESULTS AND DISCUSSION

Numerical results are presented using effective properties of carbon nanotubes. The following geometrical and material properties are used.

$$\rho = 2300\,kg/m^3,\ E = 1000\,Gpa,\ \nu = 0.19,\ G = 420Gpa,\ \ d = 1\times10^{-9}m,$$
$$h = 0.34\times10^{-9}m,\quad A = 7.85\times10^{-19}m^2,\quad I = \frac{\pi d^4}{64} = 4{,}91\times10^{-38}m^4,$$
$$ks = 0.877,\quad \Omega_0 = EI/ks\,AG,\quad a = 1.5\times10^{-9}m,\quad L = 20\,a,$$

Based on the above mathematical formulations, various type of boundary conditions can be considered by simply choosing the considered stiffness constants $K_r^L\ K_t^L, K_r^R\ K_t^R$. In table 1, the first four eigenvalues associated to different values of the nonlocal parameters e_0a/L and various stiffness's of the translational and rotational springs for slender CNT ($d/L=10^{-4}$) are presented. These results are compared to those obtained by Lu et al. [14] for clamped and clamped-free CNT. For this slender CNT the Bernoulli and Timoshenko models lead to the same results. In table 2, short CNT (d/L=0.1) is considered and the associated first fourth natural frequencies are given. These frequencies are decreasing by increasing the small scale effect e_0a/L except for clamped free boundary conditions. For C-F CNT, complex eigenvalues are obtained at critical values of the scale length e_0a/L. For the sake of clarity, curves are presented for some cases. Figure 2 shows the frequency parameters corresponding to the first to the 22th eigenmodes for a cantilever CNT with respect to the nonlocal parameter e_0a/L. It can be seen that the frequency parameters decrease by increasing of the nonlocal parameter e_0a/L and increase by increasing of the mode number. Exception is on the first order frequency parameter of cantilever CNT which is shown to be slightly increasing with e_0a/L. It is found from this figure that the nonlocal parameters affect greatly the dynamic properties of the cantilever CNT. Therefore, a reasonable choice of the value of the parameter e_0a/L is crucial to assure the validity of the nonlocal model and the vibration stability of the nanostructure. This shows that the instability at higher modes can occur very early than at the first and second with respect to the small scale parameter. It can be seen that the 1st and 2nd modes shapes coalesce when $e_0a/L=0.62$, and the 3tr and 4th coalesce when $e_0a/L=0.42$. Figures 3-a, 3-b show the variation of the modes 1 and 2 (a) modes 3 and 4 (b) of a C-F CNT at various nonlocal parameters e_0a/L. Figure 4 shows the variations of the eigenfrequency parameter associated to the first fourth modes shapes for a CNT with the nonlocal parameter e_0a/L for various translational and rotational springs. The coalescence of pair eigenmodes is addressed at critical length scales and these critical values are decreased by increasing the mode number and increased while we get far from cantilever conditions in term of the translational and rotational springs $K_t^R and\ K_r^R$.

4. CONCLUSION

In this paper, the effect of the small scale length parameter on the vibration frequencies and associated mode shapes of single walled CNTs with arbitrary boundary conditions is addressed. In order to take into account the shear effect the Timoshenko model is used. The generalized boundary conditions are considered to account for a more realistic and a large wide of boundary conditions. It is demonstrated that the small scale length parameter has a prominent effect and particularity for clamped-free CNT.

The C-F CNT will flutter at critical values of e_0a/L. This instability limit can be used as a limit for prediction values of the small length scale. This unstable behavior may also be observed for other non classical boundary conditions. Results show that when the nonlocal effect increases the frequencies decrease or coalesce for certain types of boundary conditions. In such case the dynamic behavior of the nanostructure is unstable.

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Table 1. The first four the eigenvalues with different nonlocal parameters e_0a/L for various stiffness's of the translational and rotational springs with $d/L=10^{-4}$

		$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10^9$ Clamped-clamped		$K_r^L = K_t^L = 10$ $K_r^R = K_t^R = 10$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10^2$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 0$ Clamped-free	
		[14]	present	present	present	present	present	[14]
$e_0a/L=0$	n=1	4.7300	4.7300	4.6205	3.8403	2.7147	1.8751	1.8751
	n=2	7.8532	7.8532	7.2846	5.8130	5.3349	4.6941	4.6941
	n=3	10.9956	10.9956	9.5661	8.6871	8.3661	7.8548	7.8548
	n=4	14.1372	14.1371	12.1480	11.7607	11.4375	10.9955	10.9955
$e_0a/L=0.2$	n=1	4.2766	4.2766	4.2294	3.7096	2.6512	1.8919	1.8920
	n=2	6.0352	6.0352	5.9242	4.9223	4.5746	4.1924	4.1925
	n=3	7.3840	7.3840	7.1855	6.3256	6.2204	6.0674	6.0674
	n=4	8.4624	8.4624	8.0462	7.5325	7.4751	7.3617	7.3617
$e_0a/L=0.4$	n=1	3.5923	3.5923	3.5809	3.3855	2.4973	1.9543	1.9543
	n=2	4.5978	4.5978	4.5690	3.9731	3.6659	3.3456	3.3456
	n=3	5.4738	5.4738	5.4499	4.7951	4.7391	4.8370	4.8370
	n=4	6.1504	6.1504	6.0832	5.5193	5.4654	5.2399	5.2399
$e_0a/L=0.5$	n=1	3.3153	3.3153	3.3089	3.1956	2.4070	2.0219	2.0219
	n=2	4.1561	4.1561	4.1357	3.6572	3.3386	2.9433	2.9433
	n=3	4.9328	4.9328	4.9210	4.3562	4.3060	_	_
	n=4	5.5213	5.5213	5.4797	4.9598	4.8945	_	_

Table 2. The first four order eigenvalues with different nonlocal parameters e_0a/L for various stiffness's of the translational and rotational springs with $d/L=10^{-1}$.

		$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10^9$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10^3$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10^2$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 10$	$K_r^L = K_t^L = 10^9$ $K_r^R = K_t^R = 0$
$e_0a/L=0$	n=1	4.6812	4.5805	3.8285	2.7098	1.8745
	n=2	7.5691	7.1117	5.6994	5.2426	4.6416
	n=3	10.2187	9.1376	8.2617	7.9987	7.5717
	n=4	12.5888	11.1295	10.7575	10.5405	10.2184
$e_0a/L=0.2$	n=1	4.2002	4.1599	3.6877	2.6436	1.8926
	n=2	5.7206	5.5609	4.8079	4.4717	4.1160
	n=3	6.6986	6.6266	5.9599	5.8713	5.7501
	n=4	7.3264	7.2451	6.7977	6.7615	6.6811
$e_0a/L=0.4$	n=1	3.5029	3.4638	3.3379	2.4885	1.9525
	n=2	4.3355	4.3185	3.8834	3.5713	3.2741
	n=3	4.9347	4.9266	4.5097	4.4402	4.4198
	n=4	5.3039	5.2915	4.9687	4.9375	4.7810
$e_0a/L=0.5$	n=1	3.2265	3.2217	3.1361	2.3934	2.0178
	n=2	3.9156	3.9037	3.5772	3.2516	2.8848
	n=3	4.4410	4.4369	4.1001	4.0220	_
	n=4	4.7584	4.7502	4.4639	4.4289	_

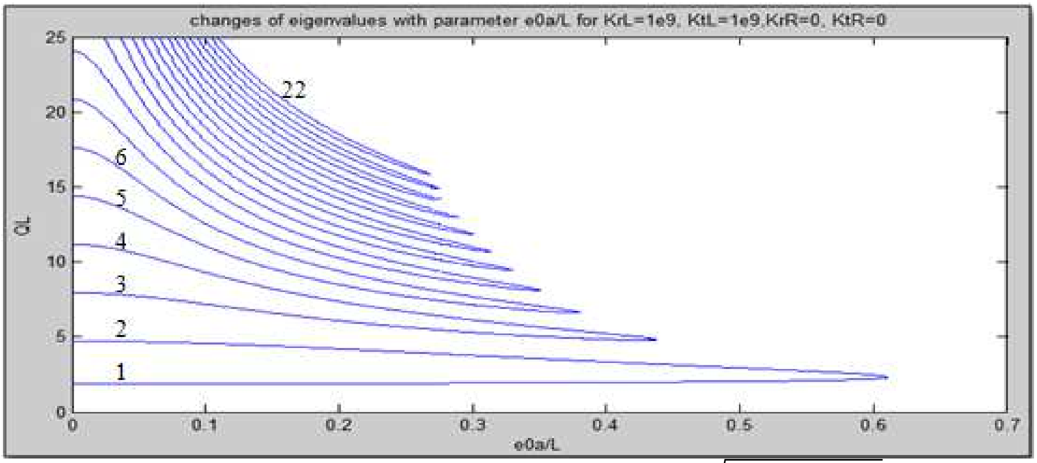

Figure 2. Small length scale (e_0a/L) effect on the frequency parameters $Q_jL = L\sqrt{\omega_j\sqrt{\rho A/EI}}$ of a cantilever single walled CNT for j=1, 22 $(K_r^L = K_t^L = 10^9,\ K_r^R = K_t^R = 0\ and\ d/L = 10^{-1})$

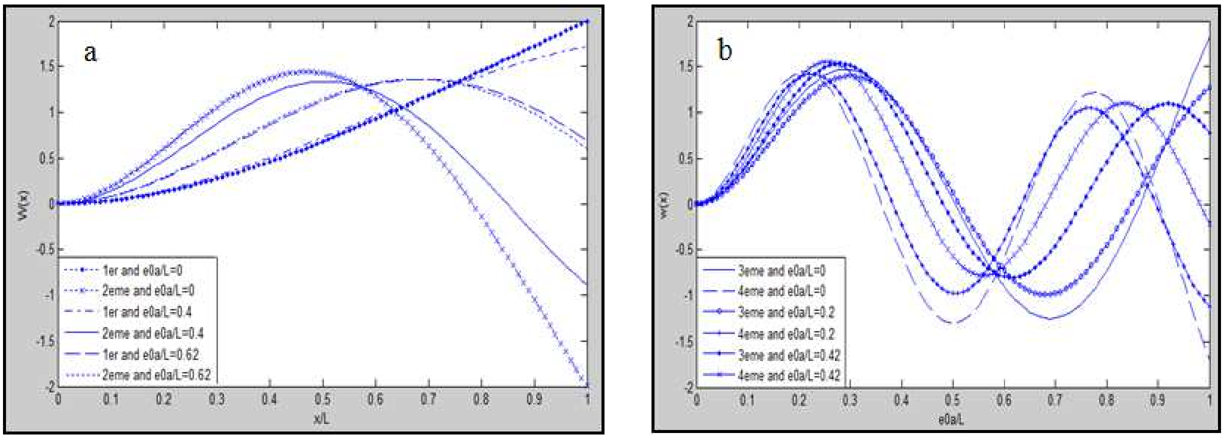

Figure 3. Modes 1 and 2 (a) modes 3 and 4 corresponding to different nonlocal parameters e_0a/L of a C-F CNT $(K_r^R and\ K_t^R = 0,\ K_r^L = K_t^L = 10^9\ and\ d/L = 10^{-1})$

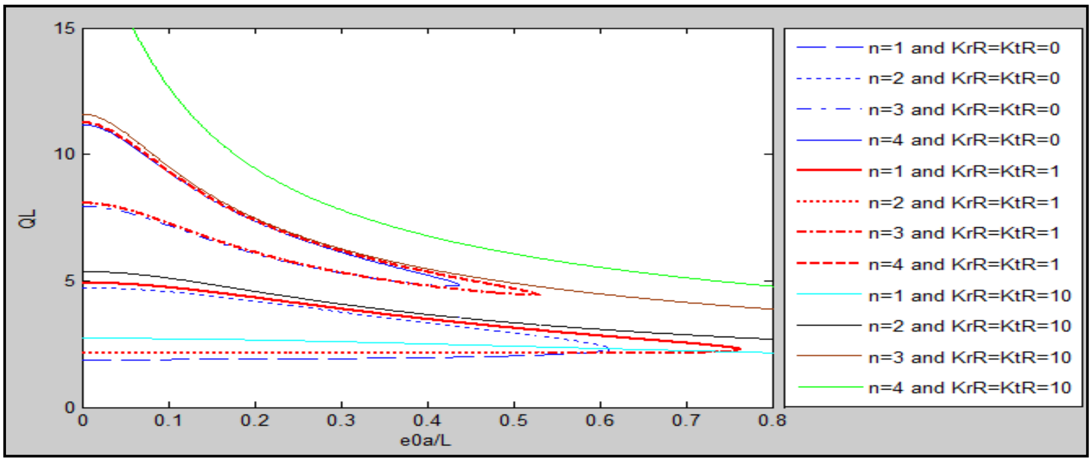

Figure 4. Variations of eigenfrequency with parameters Q_iL with respect to e_0a/L associated to various values of $K_r^R and\ K_t^R$ of a SWCNT with $(K_r^L = K_t^L = 10^9,\ d/L = 10^{-1})$