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Homogenization of an incompressible viscous fluid in crossflow filtration through thin porous layers

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Abstract
We describe the asymptotic behavior of an incompressible viscous fluid in crossflow filtration from a free domain through a thin porous layer. This porous layer has a small thickness and consists of thin channels periodically distributed. We prove that the limit problem is a Stokes system in the free domain coupled with a Darcy equation and some boundary conditions which involve an extra pressure and two second-order tensors and posed on the lower part of the boundary of the free domain. We use Γ-convergence and two-scale convergence tools for the description of this asymptotic behaviour.

Résumé
Nous décrivons le comportement asymptotique d'un fluide visqueux et incompressible circulant à partir d'un domaine libre et à travers une couche fine poreuse. Cette couche poreuse a une épaisseur faible et consiste en de fins canaux répartis périodiquement. Nous prouvons que le problème limite est un écoulement de Stokes dans le domaine libre, couplé à une loi de Darcy et des conditions aux limites posées sur la base du domaine qui impliquent une vitesse et une pression supplémentaires et deux tenseurs d'ordre deux. Le problème limite est obtenu en utilisant des méthodes de Γ-convergence et de convergence à double-échelle.

1. INTRODUCTION

The interactions between free and porous flows in different contexts have been studied by many authors, among which [1,3,4,8,9,11,15] and the references therein. A general contact law occurring in porous media between large and thin pores has been derived in [7].
The purpose of the present work is to describe the asymptotic behaviour of an incompressible viscous fluid flow occuring in a domain consisting of a free fluid domain, of a thin porous layer and of the permeable interface between them. The problem consists to find the effective contact law on the former interface which builds a boundary piece of the free fluid domain when the parameter associated to the thinness of the porous layer converges to 0.

The combination between free fluid flows and flows through thin porous layers occurs in a wide range of fluid processes, such as membrane filtration, viscous flow over a bed of solid particles and many other examples (see for instance [14]).
Let Ω be a bounded, smooth and open subset of $\mathbb{R}^3$ such that Ω is contained in $\{x \in \mathbb{R}^3 \mid x_3 > 0\}$ and $\Sigma_0 = \overline{\Omega} \cap \{x_3 = 0\}$ is non void. Ω represents the free fluid part of the domain. We define $\Gamma = \partial\Omega - \overline{\Sigma_0}$, the remaining part of the boundary of Ω. Let ε and a_ε be two parameters satisfying $0 < a_\varepsilon < \varepsilon$ and $\lim_{\varepsilon \to 0} (a_\varepsilon / \varepsilon) = 0$. We define (see Fig. 1 below)

$$\begin{cases} O &= \{x \in \mathbb{R}^3 \mid x' = (x_1, x_2) \in \Sigma_0, x_3 \in \left]-1, 0\right[\}, \\ O_\varepsilon &= \Sigma_0 \times \left]-\varepsilon, 0\right[, \\ \Sigma_\varepsilon &= \Sigma_0 \times \{-\varepsilon\}, \\ S_\varepsilon &= \partial\Sigma_0 \times \left]-\varepsilon, 0\right[, \\ \Omega_\varepsilon &= \Omega \cup \Sigma_0 \cup O_\varepsilon. \end{cases}$$

Let Y' be the unit square $\left]-1/2, 1/2\right[^2$ of $\mathbb{R}^2$ and $Y_{\varepsilon,m}$ be the ε-cell $Y_{\varepsilon,m} = Y'_{\varepsilon,m} \times \left]-\varepsilon, 0\right[$, with $Y'_{\varepsilon,m} = \varepsilon Y' + \varepsilon m$, for every $m = (m_1, m_2) \in \mathbb{Z}^2$. We define the unit cube $Z = \left]-1/2, 1/2\right[^2 \times \left]-1, 0\right[$ of $\mathbb{R}^3$ and decompose $Z = Z^1 \cup \Lambda \cup Z^2$, where Z^1 and Z^2 are open (Z^1 being connected) and separated by a smooth surface Λ, according to Fig. 1.

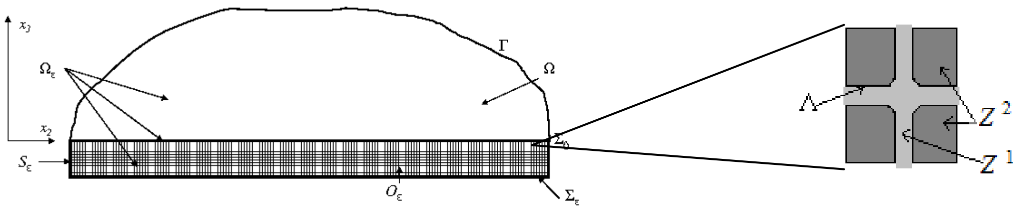

Fig. 1: A 2D cut of the domain and one enlarged a_ε-cell inside the thin porous layer O_ε.

We define

$$\begin{cases} Z_{\varepsilon,l} &= a_\varepsilon Z + a_\varepsilon l & \forall l = (l_1, l_2, l_3) \in \mathbb{Z}^3, l_3 \le 0, \\ Z^i_{\varepsilon,l} &= a_\varepsilon Z^i + a_\varepsilon l & i = 1, 2, \\ O^i_\varepsilon &= O_\varepsilon \cap \left(\cup_l Z^i_{\varepsilon,l}\right) & i = 1, 2, \\ \omega^i_\varepsilon &= \partial O^i_\varepsilon \cap \Sigma_0 & i = 1, 2, \\ \Lambda_\varepsilon &= O_\varepsilon \cap \left(\cup_l (a_\varepsilon \Lambda + a_\varepsilon l)\right), \\ \Pi_\varepsilon &= \Omega \cup \omega^1_\varepsilon \cup O^1_\varepsilon. \end{cases}$$

We observe that, for every $m \in I_\varepsilon = \{m \in \mathbb{Z}^2 \mid Y_{\varepsilon,m} \subset O_\varepsilon\}$, there exists a subset $J_{\varepsilon,m} \subset \mathbb{Z}^2 \times \mathbb{Z}^-$ such that $Y_{\varepsilon,m} = \cup_{l \in J_{\varepsilon,m}} Z_{\varepsilon,l}$. The set O_ε presents one period with ε-cells $Y_{\varepsilon,m}$ (the number of which

is of order $|\Sigma_0|/\varepsilon^2$) and one period with a_ε-cells $Z_{\varepsilon,l}$ (the number of which is of order $(\varepsilon/a_\varepsilon)^3$ in every $Y_{\varepsilon,m}$). O_ε^1 (resp. O_ε^2, Λ_ε) represents the fluid (resp. solid, interface) part of O_ε.
We consider the steady state flow of an incompressible fluid in the domain Π_ε which is the union of the free fluid part Ω, of the thin channels O_ε^1 inside the thin layer O_ε and of the interface ω_ε^1 between them. We indeed consider the steady Stokes problem for an incompressible fluid which perfectly adheres to the boundary of the domain Π_ε

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon & = & f & \text{in } \Omega, \\ -\mu\varepsilon(a_\varepsilon)^2 \Delta u_\varepsilon + \nabla p_\varepsilon & = & g & \text{in } O_\varepsilon^1, \\ \operatorname{div}(u_\varepsilon) & = & 0 & \text{in } \Pi_\varepsilon, \\ u_\varepsilon & = & 0 & \text{on } \partial\Pi_\varepsilon, \end{cases} \tag{1}$$

where μ is the viscosity of the fluid (here chosen with a constant density equal to 1).
The source terms f and g are supposed to respectively belong to $L^2(\Omega;\mathbb{R}^3)$ and $C^0(\overline{O_\varepsilon};\mathbb{R}^3)$. Motivation for different scalings in the free fluid and in the porous layer comes from the observation of the characteristic numbers: Reynolds' and Froude's numbers. We suppose that the ratio between Reynolds' and Froude's numbers, which is proportional to L^2/U, where L is the characteristic length of the porous layers O_ε^1 and U its characteristic velocity, is of the order $\varepsilon(a_\varepsilon)^2$.
As a consequence of the homogeneous Dirichlet boundary condition on ∂O_ε^1, we can extend u_ε by 0 on the solid parts of O_ε. We will keep the same notation for this extension. Introducing the function space $V_\varepsilon = \{u \in H^1(\Pi_\varepsilon;\mathbb{R}^3) \mid \operatorname{div}(u) = 0 \text{ in } \Pi_\varepsilon, u = 0 \text{ on } \partial\Pi_\varepsilon\}$, the Stokes problem (1) has a unique solution $(u_\varepsilon, p_\varepsilon)$ in $V_\varepsilon \times L^2(\Pi_\varepsilon)/\mathbb{R}$ (see [17], for example).
The main goal of this work is to describe the asymptotic behaviour of the solution $(u_\varepsilon, p_\varepsilon)$ of (1), when ε converges to 0. The limit problem is described in Corollary 9, assuming that the critical ratio $\gamma = \lim_{\varepsilon\to 0}(a_\varepsilon/\varepsilon^2)$ is finite. This limit problem involves two second-order tensors, whose coefficients involve the solutions of "local" problems, and contains a contact law of Beavers-Joseph-Saffman type (see [3,15]) on the boundary Σ_0 of the free fluid domain Ω.
For the proof of this convergence result, we use Γ-convergence tools (we refer the reader to [2] and [5] for a complete description and the properties of this variational convergence) combined with two-scale convergence arguments as described in [12].

2. A PRIORI ESTIMATES

Let us first recall some classical results concerning the div and curl operators.
Lemma 1. *There exists a linear operator* $\beta_\varepsilon : L^2(O_\varepsilon) \to H_0^1(O_\varepsilon;\mathbb{R}^3)$ *such that, for every* $h \in L^2(O_\varepsilon)$ *satisfying* $\int_{O_\varepsilon} h\,dx = 0$, *one has*

$$\begin{cases} \mathrm{div}(\beta_\varepsilon(h)) = h & \textit{almost everywhere (a.e.) in } O_\varepsilon, \\ \|\nabla\beta_\varepsilon(h)\|_{L^2(O_\varepsilon;\mathbb{R}^9)} \leq \dfrac{C(O)}{\varepsilon}\|h\|_{L^2(O_\varepsilon)}, \end{cases}$$

where $C(O)$ *is a constant which only depends on* O.

The proof of this result is performed using a change of variables in order to work in a fixed domain and proving a similar estimate in this fixed domain.

We also have the following result (see [17], for example).

Lemma 2. *Let* $D \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz continuous boundary* ∂D *and* $u \in L^2(D;\mathbb{R}^3)$ *satisfying* $\mathrm{div}(u) = 0$ *in* D. *There exists a function* $\hat{u} \in H^1(D;\mathbb{R}^3)$ *satisfying:* $div(\hat{u}) = 0$ *in D,* $curl(\hat{u}) = 0$ *in D,* $\hat{u}.n = 0$, *on* ∂D *and* $\|\hat{u}\|_{H^1(D;\mathbb{R}^3)} \leq C(D)\|u\|_{L^2(D;\mathbb{R}^3)}$.

When D is a ball $B(r)$ of radius r, this result is still valid with a constant C independent of r, see [17].

We now prove uniform estimates on the solution u_ε of (1).

Lemma 3. *One has the estimates:*

$$\sup_{\varepsilon>0}\left(\int_\Omega |\nabla u_\varepsilon|^2 dx + \varepsilon(a_\varepsilon)^2 \int_{O_\varepsilon^1} |\nabla u_\varepsilon|^2 dx\right) < +\infty,$$
$$\sup_{\varepsilon>0}\left(\int_\Omega |u_\varepsilon|^2 dx\right) < +\infty,$$
$$\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_{O_\varepsilon^1} |\varepsilon u_\varepsilon|^2 dx\right) < +\infty.$$

The proof of these estimates is obtained using the previously indicated estimates and Poincaré's and Cauchy-Schwarz' inequalities. For example, in O_ε^1, one has Poincaré's inequality:

$$\forall u \in H^1_{\Lambda_\varepsilon}(O_\varepsilon^1) : \int_{O_\varepsilon^1} |u|^2 dx \leq C(a_\varepsilon)^2 \int_{O_\varepsilon^1} |\nabla u|^2 dx.$$

where $H^1_{\Lambda_\varepsilon}(O_\varepsilon^1)$ means the set of functions of $H^1(O_\varepsilon^1)$ which vanish on Λ_ε.

Let us deal with the pressure p_ε. We first define the extension $\tilde{p}_\varepsilon$ of the pressure p_ε through $\tilde{p}_\varepsilon = p_\varepsilon$ in $\Omega \cup O_\varepsilon^1$ and $\tilde{p}_\varepsilon = \dfrac{1}{|Z^1|}\int_{Z^1} p_\varepsilon(a_\varepsilon z + a_\varepsilon l)dz$, in $Z^2_{\varepsilon,l}$ for every $l \in \mathbb{Z}^3$ such that $Z_{\varepsilon,l} \cap O_\varepsilon$ is non void. We define the zero mean-value pressure $\overline{p_\varepsilon}$ by $\overline{p_\varepsilon} = \tilde{p}_\varepsilon - \dfrac{1}{|\Omega|}\int_\Omega \tilde{p}_\varepsilon dx$ in Ω and $\overline{p_\varepsilon} = \tilde{p}_\varepsilon - \dfrac{1}{|O_\varepsilon|}\int_{O_\varepsilon} \tilde{p}_\varepsilon dx$ in O_ε.

Lemma 4. *We have the following estimates on the zero mean-value pressure* $\overline{p_\varepsilon}$ *:*

$$\sup_{\varepsilon>0}\|\overline{p_\varepsilon}\|_{L^2(\Omega)} < +\infty ; \sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_{O_\varepsilon}(\overline{p_\varepsilon})^2 dx\right) < +\infty.$$

For the proof, we again use Cauchy-Schwarz' and Poincaré's inequalities in Ω and the above construction of $\beta\left(\overline{p_\varepsilon}\right)$. We also use the existence of a restriction operator R_ε from $H_0^1(O_\varepsilon)$ to $H_0^1(O_\varepsilon^1)$ such that
$\forall v \in H_0^1(O_\varepsilon):\ \|R_\varepsilon v\|_{L^2(O_\varepsilon^1)} + a_\varepsilon \|\nabla R_\varepsilon v\|_{L^2(O_\varepsilon^1;\mathbb{R}^3)} \le C\left(\|v\|_{L^2(O_\varepsilon)} + a_\varepsilon \|\nabla v\|_{L^2(O_\varepsilon;\mathbb{R}^3)}\right)$.

3. DEFINITION OF THE LOCAL PROBLEMS

We introduce the Z^1-periodic solution (w^i, π^i) of the Stokes problem

$$\begin{cases} -\Delta w^i + \nabla \pi^i &= e_i \quad \text{in } Z^1, \\ \operatorname{div}(w^i) &= 0 \quad \text{in } Z^1, \\ w^i &= 0 \quad \text{on } \Lambda, \end{cases} \tag{2}$$

$i = 1,2,3$, where $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ denotes the i^{th} vector of the canonical basis of $\mathbb{R}^3$, δ_{ij} being Kronecker's symbol. This problem (2) has a unique solution $(w^i, \pi^i) \in H^1(Z^1;\mathbb{R}^3) \times L^2(Z^1)/\mathbb{R}$.
Let K be the second-order tensor defined through

$$K_{ij} = \int_{Z^1} \nabla w^i \cdot \nabla w^j dz = \int_{Z^1} (w^i)_j\, dz. \tag{3}$$

K is a 3×3 symmetric and positive-definite matrix (see [16]).
We also introduce the Y'-periodic solution $(w^{i,bl}, \pi^{i,bl})$, $i = 1,2,3$, of the thin layer problem

$$\begin{cases} -\Delta w^{i,bl} + \nabla \pi^{i,bl} &= 0 & \text{in } Y' \times (0,+\infty), \\ \operatorname{div}(w^{i,bl}) &= 0 & \text{in } Y' \times (0,+\infty), \\ w^{i,bl}(y',0) &= w^i(y',0) - e_3 \int_{Y'\times\{0\}} (w^i)_3 dy' & \text{on } Y' \times \{0\}. \end{cases} \tag{4}$$

The problem (4) has a unique solution $(w^{i,bl}, \pi^{i,bl}) \in V^{bl} \times L^2_{loc}(Y' \times]0,+\infty[)$ with

$$V^{bl} = \left\{ \begin{array}{l} w \in L^2_{loc}(Y' \times]0,+\infty[;\mathbb{R}^3) \mid \nabla w \in L^2(Y' \times]0,+\infty[;\mathbb{R}^9), \\ \quad \operatorname{div}(w) = 0 \ in\ Y' \times]0,+\infty[,\ w\ Y'\text{-}\ periodic \end{array} \right\}.$$

This solution $(w^{i,bl}, \pi^{i,bl})$ satisfies Phragmen-Lindelöf type inequalities, as described in the following result.

Lemma 5. *(see [8,10,13]). For every a, b in* $\mathbb{R}$, *with* $b > a > 0$, *we define the strip* $\omega(a,b) = \{ y \in Z^+ \mid y' \in Y', a < y_3 < b \}$. *There exist two positive constants* c_0 *and* c_1, *a constant vector* $c^i = (c_1^i, c_2^i, c_3^i) \in \mathbb{R}^3$ *and a real constant* c_π^i *such that, for every nonnegative integer s*
$\|w^{i,bl} - c^i\|_{L^2(\omega(s,s+1);\mathbb{R}^3)} \le c_0 e^{-c_1 s}$; $\|\pi^{i,bl} - c_\pi^i\|_{L^2(\omega(s,s+1))} \le c_0 e^{-c_1 s}$; $\|\nabla w^{i,bl}\|_{L^2(\omega(s,s+1);\mathbb{R}^9)} \le c_0 e^{-c_1 s}$.

We define the 3×3 symmetric and positive-definite matrix K^{bl} associated to the thin layer $\Sigma_0\times]0,\varepsilon[$ through

$$K_{ij}^{bl}=\int_{Y'\times(0,+\infty)}\nabla w^{i,bl}\cdot\nabla w^{j,bl}dy\,. \qquad (5)$$

4. CONVERGENCE

4.1 Preliminaries

Lemma 6. One has:

1) *Let* $v_\varepsilon\in L^2(O_\varepsilon;\mathsf{R}^3)$ *be such that* $\sup_\varepsilon\left(\int_{O_\varepsilon}|v_\varepsilon|^2dx/\varepsilon\right)<+\infty$. *There exists* $v_0\in L^2(\Sigma_0;\mathsf{R}^3)$ *such that, up to some subsequence*

$$\int_{O_\varepsilon}v_\varepsilon\cdot\varphi\frac{dx}{\varepsilon}\underset{\varepsilon\to 0}{\to}\int_{\Sigma_0}v_0(x')\cdot\varphi(x',0)dx',\ \forall\varphi\in C^0(\mathsf{R}^3;\mathsf{R}^3).$$

2) *For every function* $\psi\in C^\infty(O;\mathsf{R}^3)$, *which is 1-periodic in the direction* x_3, *one has the following two-scale convergence*

$$\int_{O_\varepsilon}v_\varepsilon\cdot\psi\left(x',\frac{x_3}{\varepsilon}\right)\frac{dx}{\varepsilon}\underset{\varepsilon\to 0}{\to}\int_{\Sigma_0}\int_{-1}^{0}v_0^*(x',s)\cdot\psi(x',s)dx'ds,$$

with $v_0(x')=\int_{-1}^{0}v_0^*(x',s)ds$.

The first point is somehow classical using Fenchel's inequality and Riesz' representation theorem. The second one is an easy consequence of the two-scale convergence (see [12]).

4.2 Main result

Thanks to the estimates given in the above Lemmas 3 and 4, and to Lemma 6, there exist $u_0\in L^2(\Omega;\mathsf{R}^3)$, $p_0\in L^2(\Omega)$, $v_0\in L^2(\Sigma_0;\mathsf{R}^3)$ and $q_0\in L^2(\Sigma_0)$ such that, up to subsequences

$$\begin{array}{llll}
(u_\varepsilon)_{|\Omega} & \underset{\varepsilon\to 0}{\to} & u_0 & w\text{-}H^1(\Omega;\mathsf{R}^3),\\
(\overline{p_\varepsilon})_{|\Omega} & \underset{\varepsilon\to 0}{\to} & p_0 & w\text{-}L^2(\Omega;\mathsf{R}^3),\\
\int_{\mathbf{O}_\varepsilon}\varepsilon u_\varepsilon\cdot\varphi\frac{dx}{\varepsilon} & \underset{\varepsilon\to 0}{\to} & \int_{\Sigma_0}v_0(x')\cdot\varphi(x',0)dx' & \forall\varphi\in C_0(\mathsf{R}^3;\mathsf{R}^3),\\
\int_{\mathbf{O}_\varepsilon}\overline{p_\varepsilon}\psi\frac{dx}{\varepsilon} & \underset{\varepsilon\to 0}{\to} & \int_{\Sigma_0}q_0(x')\psi(x',0)dx' & \forall\psi\in C_0(\mathsf{R}^3).
\end{array} \qquad (6)$$

Furthermore, one has: $\mathrm{div}(v_0)=(v_0)_3=(u_0)_3=0$, *in* Σ_0.

From the estimates of Lemmas 3 and 4 and to (6), we deduce the appropriate rescaling of the solution in the thin layer O_ε and the appropriate notion of convergence for the problem (1) under consideration.

Definition 7. *For every* $u\in H_0^1(\mathsf{R}^3;\mathsf{R}^3)$, *we define the rescaled solution* $r^\varepsilon(u)=\varepsilon u\mathbf{1}_{O_\varepsilon}$, *inside the thin layer* O_ε.

Let $U_\varepsilon \in V_\varepsilon$ *, for every ε. The sequence* $\left(U_\varepsilon, r^\varepsilon(U_\varepsilon)\right)_\varepsilon$ τ *-converges to* $(U,V) \in W_0$ *, with*

$$W_0 = \left\{ \begin{array}{c} (u,v) \in H^1(\Omega;\mathbb{R}^3) \times L^2(\Sigma_0;\mathbb{R}^3) \mid \operatorname{div}(u) = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma, u_3 = 0 = v_3 \text{ on } \Sigma_0 \\ \operatorname{div}(v) = 0 \text{ in } \Sigma_0, v \cdot n = 0 \text{ on } \partial\Sigma_0 \end{array} \right\},$$

if the following properties are satisfied

$$\begin{array}{lll} U_\varepsilon & \underset{\varepsilon \to 0}{\rightarrow} U & w\text{-}H^1(\Omega;\mathbb{R}^3), \\ \sup_\varepsilon \left(\dfrac{1}{\varepsilon} \int_{\mathbb{R}^3} \left| r^\varepsilon(U_\varepsilon) \right|^2 dx \right) & < +\infty, & \\ \sup_\varepsilon \left(\varepsilon (a_\varepsilon)^2 \int_{O_\varepsilon} \left| \nabla U_\varepsilon \right|^2 dx \right) & < +\infty, & \\ \dfrac{1}{\varepsilon} \int_{\mathbb{R}^3} r^\varepsilon(U_\varepsilon) \cdot \varphi dx & \underset{\varepsilon \to 0}{\rightarrow} \int_{\Sigma_0} V(x') \cdot \varphi(x',0) dx' & \forall \varphi \in C^0(\mathbb{R}^3;\mathbb{R}^3). \end{array}$$

Our main result is the following.

Theorem 8. *Let* F_ε *be the energy functional associated to (1), that is defined in the space* $L^2(\Pi_\varepsilon;\mathbb{R}^3)$ *through*

$$F_\varepsilon(u) = \begin{cases} \mu \int_\Omega |\nabla u|^2 dx + \mu \varepsilon (a_\varepsilon)^2 \int_{O_\varepsilon^1} |\nabla u|^2 dx & \text{if } u \in V_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

The sequence $(F_\varepsilon)_\varepsilon$ Γ*-converges in the above-defined topology* τ *to the functional* F_0 *defined on* $L^2(\Omega;\mathbb{R}^3) \times L^2(\Sigma_0;\mathbb{R}^3)$ *through*

$$F_0(u,v) = \begin{cases} \mu \int_\Omega |\nabla u|^2 dx + \mu \int_{\Sigma_0} K^{-1} v \cdot v dx' & \\ \quad + \gamma \mu \int_{\Sigma_0} K^{bl} \left(u|_{\Sigma_0} - v \right) \cdot \left(u|_{\Sigma_0} - v \right) dx' & \text{if } (u,v) \in W_0 \\ +\infty & \text{otherwise,} \end{cases}$$

where K *and* K^{bl} *are the second-order tensors respectively defined in (3) and (5).*

The proof of this result is quite long and will be omitted. It can be decomposed in three steps. We first build appropriate test-functions. This step is quite tedious and we refer the reader to the paper [7] for the proof of a quite similar result. Then we verify the lim sup and the lim inf conditions of the Γ-convergence within this context (see [2,5]).

We deduce from this main theorem the following convergence result.

Corollary 9. *Assume* $\gamma = \lim_{\varepsilon \to 0} (a_\varepsilon / \varepsilon^2) \in \left]0, +\infty\right[$*. The sequence* $\left(u_\varepsilon, r^\varepsilon(u_\varepsilon)\right)_\varepsilon$*, where* u_ε *is the solution of (1), converges in the topology* τ *to the solution* (u_0, v_0) *of the minimization problem* $\min_{(u,v) \in W_0} \left(F_0(u,v) - L(u,v) \right)$*, with* $L(u,v) = \int_\Omega f \cdot u dx - \int_{\Sigma_0} g(.,0) \cdot v dx'$.

Up to some subsequence, the sequence $\left(\left(\overline{p_\varepsilon} \right)|_\Omega \right)_\varepsilon$ *converges, in the weak topology of* $L^2(\Omega)$*, to some limit pressure* p_0 *which satisfies* $\int_\Omega p_0 dx = 0$*. Moreover, there exists* $q_0 \in L^2(\Sigma_0)$ *such that* $\int_{\Sigma_0} q_0 dx' = 0$ *and*

$$\forall \varphi \in C^0(\mathbb{R}^3) : \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} \int_{O_\varepsilon} \overline{p_\varepsilon} \varphi dx \right) = \int_{\Sigma_0} q_0(x') \varphi(x',0) dx'.$$

(u_0, p_0) *is the solution of the limit problem*

$$\begin{cases} -\mu\Delta u_0 + \nabla p_0 & = & f & \text{in } \Omega, \\ \operatorname{div}(u_0) & = & 0 & \text{in } \Omega, \\ u_0 & = & 0 & \text{on } \Gamma, \\ (u_0)_3 & = & 0 & \text{on } \Sigma_0, \\ \dfrac{\partial u_0}{\partial x_3}|_{x_3=0} & = & -\gamma K^{bl}\left((u_0)|_{\Sigma_0} - v_0\right) & \text{on } \Sigma_0, \\ \gamma\mu K^{bl}\left((u_0)|_{\Sigma_0} - v_0\right) - \mu K^{-1} v_0 & = & \nabla q_0 - (g_1, g_2, 0) & \text{on } \Sigma_0, \\ \operatorname{div}(v_0) & = & 0 & \text{on } \Sigma_0, \\ v_0 \cdot n & = & 0 & \text{on } \partial\Sigma_0, \end{cases}$$

with the further velocity v_0 *and the further pressure* q_0 *on* Σ_0*, defined in (6), where* K^{bl} *is the boundary layer tensor defined in (5),* K *is the permeability tensor defined in (3).*

Remark 10. From the above result and from the properties of the Γ-convergence, we can deduce the asymptotic behaviours of u_ε in the cases $\gamma = +\infty$ or $\gamma = 0$, mainly using comparison arguments and the properties of the Γ-convergence.

5. CONCLUSION

We described the asymptotic behavior of an incompressible viscous fluid in crossflow filtration from a free domain through a thin porous layer. This porous layer has a small thickness and contains thinner channels which are periodically distributed. The limit law consists of a Stokes system in the free fluid coupled to a Darcy equation on the former interface, through an interfacial condition of Beavers-Joseph-Saffmann type **[**3**]**.

In their paper **[**3**]**, Beavers and Joseph concluded experimentally that the difference between the slip velocity of the free fluid and the tangential component of the seepage velocity at the interface is proportional to the shear from the free fluid: $\sqrt{K}\dfrac{\partial v_\tau^1}{\partial n} = \gamma\left(v_\tau^1 - v_\tau^2\right)$, where n is the outer unit normal, v_τ^1 is the tangential component of the free fluid velocity, v_τ^2 is the tangential component of the Darcy velocity of the fluid in the porous medium, K is the permeability of the porous medium and γ is some slippage coefficient, which depends on the geometrical structure of the porous medium. This model has been then studied by Saffman from a theoretical point of view in **[**15**]**, who proved that the seepage velocity contribution can be dropped.

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