Vol. **2**, 4. 431-440 (2011)

Revue de Mécanique Appliquée et Théorique

Taylor-SPH method for shock wave propagating in viscoplastic continua

Mokhtar Mabssout
Equipe Matériaux et Mécanique des Structures, Faculté des Sciences et Techniques - BP : 416, Tanger- Maroc
Auteur correspondant : mabssout@hotmail.com

María Isabel Herreros
Escuela Técnica Superior de Ingeniería Informática, Universidad Rey Juan Carlos (URJC), Madrid - Espagne

Abstract
In this article, a new time integration algorithm using a corrected SPH spatial discretization is presented. The equations are written in terms of stress and velocity. To avoid numerical instabilities, two different sets of particles are used for time integration and a corrected Lagrangian kernel has been used for the spatial approximation. This new algorithm is applied to solve the propagation of shock waves in viscoplastic media. The results are compared with those obtained with the Finite Element Method. The proposed method is shown to be stable, robust and it provides solutions of good accuracy.
Keywords: Taylor-SPH; Taylor-Galerkin; Shock wave; Viscoplastic; FEM

Résumé
Dans cet article, on présente un nouvel algorithme d'intégration dans le temps associé à la méthode sans maillage SPH pour la discrétisation dans l'espace. Les équations du modèle sont écrites en fonction des variables vitesse - contrainte. Pour éviter les instabilités numériques, deux groupes de particules ont été utilises pour la discrétisation dans le temps et une fonction noyau corrigée type Lagrangienne a été utilisée pour la discrétisation dans l'espace. Cette nouvelle méthode a été utilisée pour résoudre le problème de propagation d'ondes de choc dans un milieu viscopalstique. Les résultats ont été comparés avec ceux obtenus par la Méthode des Eléments Finis. L'algorithme proposé est stable, robuste et précis.
Mots-clés: Taylor- SPH; Taylor-Galerkin; Onde de choc; Viscoplastique; MEF

1. INTRODUCTION

Mesh based numerical methods such as Finite Element Method (FEM), Finite Difference Method (FDM) and Finite Volume Method (FVM) have been widely used to solve many engineering problems in Solid and Fluid Mechanics. Despite of their great success, mesh based numerical methods suffer from some difficulties when dealing with problems where large deformations, discontinuities and crack propagation are involved. To overcome these difficulties, some meshfree methods were developed in the past. One example of meshfree method is the Smoothed Particle Hydrodynamics (SPH). The SPH method was developed in 1977 by Lucy [6] and Gingold and Monaghan [4]. The SPH method presents several difficulties when dealing with Dynamics and shock wave propagation. The authors of the present paper have published in previous works some different alternatives to solve the propagation of shock waves in solids using FEM [7] [8]. In this paper, a new time discretization algorithm using SPH for solving the propagation of shock waves in viscoplastic media is developed. It uses a two-steps time discretization algorithm by means of a Taylor series expansion and a corrected

SPH method for the spatial discretization. In order to avoid numerical instabilities, two different sets of particles are considered for the time discretization and a Lagrangian kernel is used for the spatial approximation [1]. Both, Lagrangian kernel and its gradient are corrected to satisfy the consistency conditions. To assess the performance of the proposed method, some numerical applications in 1D and 2D are given. The results are compared to analytical solutions and to the numerical results obtained with FEM.

The paper is organized as follows. First the governing equations for dynamic problems in viscoplastic media are given in terms of stress and velocity. In section 3, numerical discretization is presented using the proposed Taylor-SPH method. To assess the performance of the proposed method, some numerical applications in 1D and 2D are described in Section 4. The results are compared to analytical solutions and to the numerical results obtained with FEM.

2. GOVERNING EQUATIONS

The balance of momentum and constitutive equations can be written in terms of stress $\boldsymbol{\sigma}$ and velocity $\mathbf{v}$ as follows:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = div\boldsymbol{\sigma} \tag{1}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{D}^e : \left(\frac{\partial \boldsymbol{\varepsilon}}{\partial t} - \frac{\partial \boldsymbol{\varepsilon}^{vp}}{\partial t} \right) \tag{2}$$

where ρ is the density, $\mathbf{D}^e$ is the elastic constitutive tensor, $\boldsymbol{\varepsilon}$ the strain tensor and ε^{vp} is the viscoplastic strain given by Perzyna's law [10]:

$$\frac{\partial \boldsymbol{\varepsilon}^{vp}}{\partial t} = \gamma \left\langle \frac{F - F_o}{F_o} \right\rangle^N \frac{\partial F}{\partial \boldsymbol{\sigma}} \tag{3}$$

where γ and N are model parameters, F_o is the yield surface and, for Von Mises material, F is taken as the effective deviatoric stress. In two dimensional situations, stress and strain are represented as

$$\begin{aligned} \sigma &= (\sigma_{11}, \sigma_{22}, \sigma_{12})^T \\ \varepsilon &= (\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12})^T \end{aligned} \tag{4}$$

The kinematic relation is written as

$$\frac{\partial \varepsilon_{ij}}{\partial t} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{5}$$

The equations (1) and (2) can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + div\, \mathbf{F} = \mathbf{S} \tag{6}$$

where $\mathbf{U}^T = (\sigma_{11} \quad \sigma_{22} \quad \sigma_{12} \quad v_1 \quad v_2)$, $\mathbf{F}_x^T = (-D_{11}v_1 \quad -D_{12}v_1 \quad -D_{33}v_2 \quad -\frac{\sigma_{11}}{\rho} \quad -\frac{\sigma_{12}}{\rho})$

$$\mathbf{F}_y^T = (-D_{12}v_2 \quad -D_{22}v_2 \quad -D_{33}v_1 \quad -\tfrac{\sigma_{12}}{\rho} \quad -\tfrac{\sigma_{22}}{\rho}),\ \mathbf{S}^T = (-D_{11}\dot{\varepsilon}_1^{vp} - D_{12}\dot{\varepsilon}_2^{vp} \quad -D_{12}\dot{\varepsilon}_1^{vp} - D_{22}\dot{\varepsilon}_2^{vp} \quad -D_{33}\dot{\varepsilon}_3^{vp} \quad 0 \quad 0)$$

being **U** the unknown vector, **F** the advective flux tensor, **S** the source vector and D_{ij} the components of the elastic matrix $\mathbf{D}^e$.

3. NUMERICAL DISCRETIZATION

To solve equation (6) using the classical SPH method, the SPH spatial discretization is applied first, obtaining a set of simultaneous ordinary differential equations with respect to time. This set of equations is then integrated in time using one of the standard techniques, such as the Runge-Kutta schemes. In the presence of discontinuities, such as shock waves, the classical SPH method presents some numerical problems, such as numerical dispersion and diffusion. To overcome these difficulties, the authors propose here an alternative method which consists of applying first the time discretization in two steps and afterwards the corrected SPH for spatial discretization.

3. 1. Two-steps time discretization

Time discretization of equation (6) is carried out using a Taylor series expansion in time of **U** up to second order:

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \left.\frac{\partial \mathbf{U}}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \mathbf{U}}{\partial t^2}\right|^n + \mathrm{O}(\Delta t)^3 \tag{7}$$

The first order time derivative of the unknowns can be calculated using equation (6)

$$\left.\frac{\partial \mathbf{U}}{\partial t}\right|^n = (\mathbf{S} - div\ \mathbf{F})^n \tag{8}$$

and the second order derivative with respect to time is given by

$$\left.\frac{\partial^2 \mathbf{U}}{\partial t^2}\right|^n = \left(\frac{\partial \mathbf{S}}{\partial t}\right)^n - div\left(\frac{\partial \mathbf{F}}{\partial t}\right)^n \tag{9}$$

First step: In order to obtain the time derivatives of fluxes and sources at time t^n, the values of the unknowns at an intermediate time $t^{n+1/2}$ must be obtained first:

$$\mathbf{U}^{n+1/2} = \mathbf{U}^n + \frac{\Delta t}{2}(\mathbf{S} - \nabla\ \mathbf{F})^n \tag{10}$$

Using the values of fluxes and sources at $\mathbf{t}^{n+1/2}$, equation (9) can be written as

$$\left.\frac{\partial^2 \mathbf{U}}{\partial t^2}\right|^n = \frac{2}{\Delta t}\left(\mathbf{S}^{n+1/2} - \mathbf{S}^n - div\ (\mathbf{F}^{n+1/2} - \mathbf{F}^n)\right) \tag{11}$$

Second step: Substituting now the expressions obtained for the first and second order time derivatives, (8) and (11), in the Taylor series expansion (7) we obtain

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} + \Delta t \left(\mathbf{S} - \nabla\, \mathbf{F}\right)^{n+1/2} \tag{12}$$

3. 2. SPH spatial discretization

Space discretization of equation (12) is carried out using a corrected SPH method. A brief summary of the method is presented herein. For more details about the SPH method and its applications one can be referred to [2] [3] [5].

3. 2. 1. Discrete approximation of functions

The kernel approximation of an arbitrary function *f* is defined by:

$$\left\langle f(x) \right\rangle = \int_{\Omega} f(x')W(x-x',h)dx' \tag{13}$$

where the brackets denote a kernel approximation, *x* and *x'* are the coordinates vectors, *W(x'-x, h)* is the kernel function and *h* is the smoothing length that defines the size of the kernel support. It has been shown by Belytschko et al. [1] that the Lagrangian kernel eliminates the tensile instability, and therefore it will be used here since it provides a more consistent procedure. Thus the Lagrangian kernel is expressed in terms of the material coordinates as $W(X-X',h_o)$ and the neighbours of influence do not change during the simulation, remaining h_o as a constant value. Several kernels have been proposed in the past, in this work the B-spline function introduced by Monaghan and Lattanzio [9] has been used:

$$W(\xi) = C \begin{cases} (\frac{2}{3} - \xi^2 + \frac{1}{2}\xi^3) & 0 \le \xi \le 1 \\ \frac{1}{6}(2-\xi)^3 & 1 \le \xi \le 2 \\ 0 & \xi \ge 2 \end{cases} \tag{14}$$

being $\xi = \frac{|X_j - X_i|}{h_o} = \frac{r}{h_o}$; the scaling factor C is given by $\frac{1}{h_o}$, $\frac{15}{7\pi h_o^2}$ and $\frac{3}{2\pi h_o^3}$ in 1D, 2D and 3D respectively. In the SPH approximation, a continuum is represented by a set of particles, thus the discrete approximation of the continuous form (13) can be written as

$$f_I = \sum_{J=1}^{N} f_J W_{IJ} \frac{m_J}{\rho_J} \tag{15}$$

where the summation subscript *J* denotes a particle label and runs over all particles, *N*, inside the domain, such that $|X_J - X_I| \le \kappa\, h_o$, being κ a positive parameter; m_J and ρ_J are the mass and density associated to particle *J* . $W_{IJ} = W(\mathbf{X}_J - \mathbf{X}_I, h_o)$ denotes the value of the kernel centred at node *I* at position *J*.

3. 2. 2. Discrete approximation of derivatives

The SPH integral representation of the derivative of a function $f(X)$ is given by

$$\langle \nabla f(X) \rangle = -\int_{\Omega} f(X') \, \nabla W(X - X', h_0) dX' \tag{16}$$

The discrete form of (16) is given by

$$\nabla f_I = \sum_{J=1}^{N} \frac{m_J}{\rho_J} (f_J - f_I) \nabla W_{IJ} \tag{17}$$

where $\nabla f_I = \langle \nabla f(X_I) \rangle$; $\nabla W_{IJ} = \frac{W'}{h_0} \frac{X_{IJ}}{|X_{IJ}|}$ and $X_{IJ} = X_I - X_J$

3. 2. 3. Corrected form of the SPH approximations

The correction of the approximating function and its derivatives are given in order to satisfy the zeroth and the first order completeness respectively.

- Corrected form of the approximating function

To fulfill the zeroth order completeness of the approximation in the whole domain Ω, the corrected particle approximation for any function f is given by

$$f_I = \sum_{J=1}^{N} f_J \tilde{W}_{IJ} \Omega_J = \frac{\sum_{J=1}^{N} f_J W_{IJ} \Omega_J}{\sum_{J=1}^{N} W_{IJ} \Omega_J} \tag{18}$$

$\tilde{W}_{IJ} = \frac{W_{IJ}}{\sum_{J=1}^{N} W_{IJ} \Omega_J}$ is the corrected Kernel·

- Corrected form of the derivatives of the approximating function

To fulfill the first order completeness, the corrected form of the derivative of the approximating function is given by

$$\nabla f_I = \mathbf{C}_I \left(\sum_{J=1}^{N} f_{IJ} \, \nabla W_{IJ} \Omega_J \right) = \sum_{J=1}^{N} f_{IJ} \, \tilde{\nabla} W_{IJ} \Omega_J \tag{19}$$

where

$$\mathbf{C}_I = \left(\sum_{J=1}^{N} X_{IJ} \otimes \nabla W_{IJ} \Omega_J \right)^{-1}, \qquad \tilde{\nabla} W_{IJ} = \mathbf{C}_I \nabla W_{IJ} \tag{20}$$

3. 2. 4. Discretized equations using the proposed method: Taylor-SPH

To perform the time discretization it will be necessary the use of an auxiliary set of particles that will be called “virtual” particles. These “virtual” particles will be interspersed among the “real” particles in a similar manner it was done in stress-point integration methods. Time discretization of model equations is carried out in two steps:

First Step: Applying the corrected SPH spatial discretization to the first step of time discretization, (10), we obtain

$$\langle \mathbf{U} \rangle_{VP}^{n+1/2} = \langle \mathbf{U} \rangle_{VP}^{n} + \left\langle \frac{\Delta t}{2} \right\rangle \left[\langle \mathbf{S} \rangle_{VP}^{n} - \langle \nabla\, \mathbf{F} \rangle_{VP}^{n} \right] \quad (21)$$

The subscript *VP* indicates that the variables are computed at the “virtual” particles positions. Using the corrected form for the approximation of functions and derivatives given by equations (18) and (19), we obtain the value of the variable **U** at $t^{n+1/2}$

$$\mathbf{U}_{VP}^{n+1/2} = \mathbf{U}_{VP}^{n} + \frac{\Delta t}{2} \left[\sum_{J=1}^{Nr} \frac{m_J}{\rho_J} \mathbf{S}_J^n \, \tilde{W}_{IJ} - \sum_{J=1}^{Nr} \frac{m_J}{\rho_J} \mathbf{F}_J^n \, \tilde{\nabla} W_{IJ} \right] \quad (22)$$

being J the “real” particles, such that $|X_J - X_{VP}| \leq 2\, h_o$.

Second Step: Applying the corrected SPH spatial discretization to equation (12), we obtain

$$\langle \mathbf{U} \rangle_{RP}^{n+1} = \langle \mathbf{U} \rangle_{RP}^{n} + \langle \Delta t \rangle \left[\langle \mathbf{S} \rangle_{RP}^{n+1/2} - \langle \nabla\, \mathbf{F} \rangle_{RP}^{n+1/2} \right] \quad (23)$$

The subscript *RP* indicates that the variables are computed at the “real” particles positions. Using expressions (18) and (19), the value of the variable **U** at t^{n+1} is given by

$$\mathbf{U}_{RP}^{n+1} = \mathbf{U}_{RP}^{n} + \Delta t \left[\sum_{J=1}^{Nv} \frac{m_J}{\rho_J} \mathbf{S}_J^{n+1/2} \, \tilde{W}_{IJ} - \sum_{J=1}^{Nv} \frac{m_J}{\rho_J} \mathbf{F}_J^{n+1/2} \, \tilde{\nabla} W_{IJ} \right] \quad (24)$$

where J are the “virtual” particles such that $|X_J - X_{RP}| \leq 2\, h_o$. In the following, the new SPH formulation presented here will be referred to as TSPH, which stands for Taylor-SPH.

4. NUMERICAL EXAMPLES

4. 1. Propagation of a shock wave on a viscoplastic 1D bar

In this section it will be solved the problem of a shock wave propagating in a viscoplastic 1D bar, using the proposed TSPH method. The problem consists of a bar of length 1 m with a unit section, which has been spatially discretized using 50 “real” particles and 49 “virtual” particles. In the case of the model parameter $N = 0$, the problem has an analytical solution and the stress at x = L is given by

$$\sigma(L,t) = \left[2v_o\sqrt{\rho E} - \frac{1}{2}E\gamma(t-t_o) \right] H\left(\frac{t-t_o}{t_r - t} \right) \tag{25}$$

where $t_o = L\sqrt{\rho / E}$; $t_r = t_o + t_f$ being t_f = 2.5 ms ; $H(t)$ is the Heaviside function and

$$v_o(t) = \begin{cases} 1 \text{ m/s} & for \quad t \le t_f = 2.5\,ms \\ 0 & for \quad t > t_f = 2.5\,ms \end{cases}$$

The material properties are: ρ=2000 Kg/m^3, E=8 10^7 Pa, yield stress $\bar{\sigma}_0 = 4\ 10^5$ Pa and softening modulus H= –E/10. The model parameters are N=0 and γ=1s^{-1}. Figure 1 presents the comparison between the analytical solution (25) at L = 1 m and the result obtained using the proposed TSPH method. It can be observed that the results obtained by the proposed method TSPH are in complete agreement with the analytical solution. Figure 2 shows a comparison between the analytical solution and the results obtained when using FEM with the Newmark scheme and the Taylor-Galerkin algorithm (TG) [7] for N = 0 and γ= 1s^{-1}. It can be observed that the numerical solutions obtained with Finite Element present oscillations. The amplitude of the oscillations is much higher when using the Newmark scheme, for which the front is also smoothed. These results confirm the better performance of the proposed method (TSPH) for shock wave propagation.

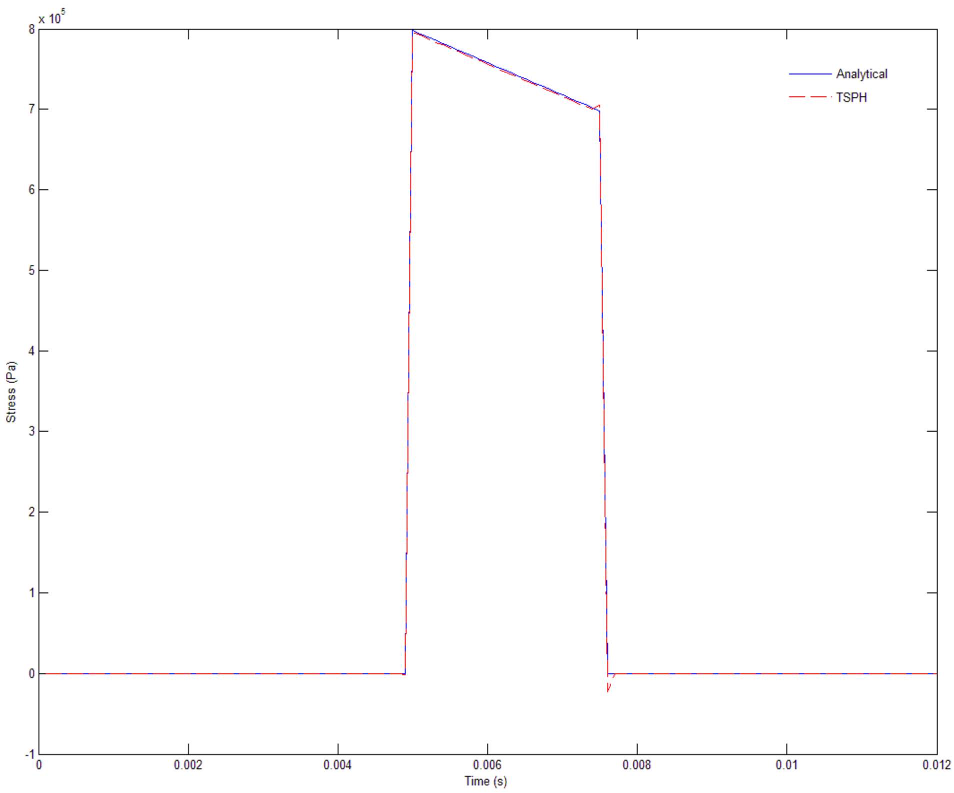


Figure 1: Stress $\sigma(x$ = 1m; $t)$ in the viscoplastic case (N = 0 and γ= 1) using TSPH

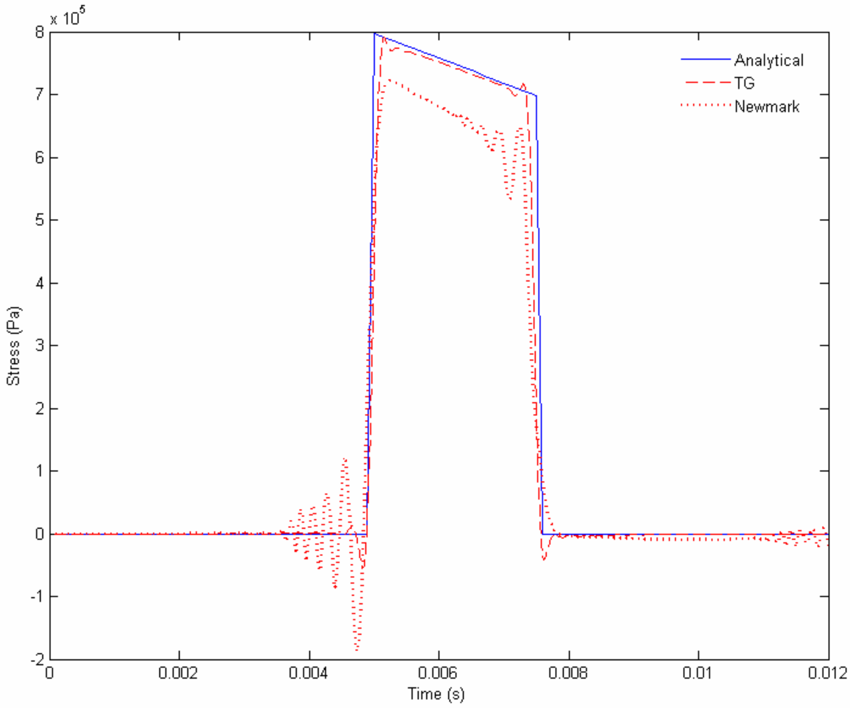


Figure 2: Stress $\sigma(x = 1\mathrm{m}; t)$ in the viscoplastic case ($N = 0$ and $\gamma = 1$) using FEM

4. 2. Bending of a cantilever beam

The purpose of this example is to show the behaviour of the proposed model in bending dominated situations. It has been chosen the simple example of a cantilever beam subjected to a vertical load at its free end F_o=*800 N*. Length and width of the beam are L = 1 m and b = 0.02 m. Material is assumed to be elastic, with Young's modulus $E = 8\ 10^9$ Pa and Poisson's ratio $\nu = 0.0$. Density has been taken as $2\ 10^3$ Kg/m^3. A structured arrangement (in three rows) of 63 “real” particles and 40 “virtual” particles is used. Figure 3 depicts the vertical displacement at the right end of the beam as a function of the increasing number of particles. It is shown that the TSPH method converges quickly to the analytical solution and only a small number of particles is required to obtain very accurate results.

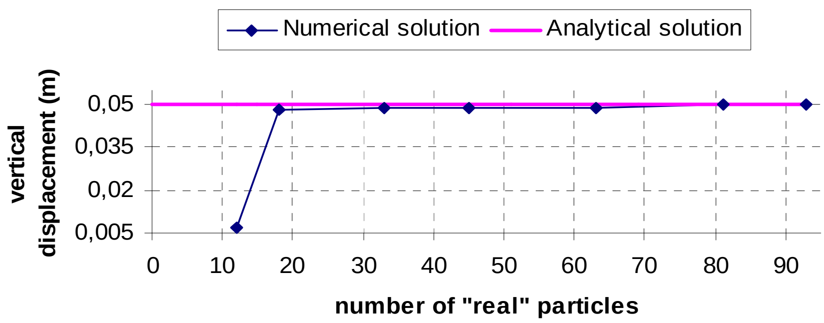


Figure 3: Deflexion at L = 1 m with an increasing number of “real” particles

4. 3. Strain localization in 2D problem

Simulation of the shear band in a two dimensional specimen subjected to a shock on its upper face (Figure 4) is analyzed here. The spatial domain consists of a square of side 1 m, but for symmetry reasons only one half will be considered in the analysis. The material behaviour is a viscoplastic with Von Mises yield surface. The parameters used in the computation are: Density ρ = 2000 Kg/m³, Young modulus E=8 10^7 Pa, Poisson's ratio ν = 0.3, initial yield stress is σ =5 10^5 Pa , softening modulus H = -E/10 and model parameters N =1; γ = 25 s^{-1}. The domain is discretized using 861 "real" particles and 800 "virtual" particles. The applied boundary conditions are the following:

(i) On Γ_1: $v_x = v_y = 0$ (ii) On Γ_2: $v_x = 0$ and $\sigma_{xy} = 0$
(iii) On Γ_3: $\sigma_{xx} = 0$ and $\sigma_{xy} = 0$ (iv) On Γ_4: $v_y = v_o(t)$

where $v_o(t)$ is given by:

$$v_o(t) = \begin{cases} 1m/s & t \le t_f = 8ms \\ 0 & t > t_f = 8ms \end{cases}$$

Figure 5 shows the viscoplastic strain contours and the deformed configuration of the "real" particles obtained with the proposed method (TSPH).

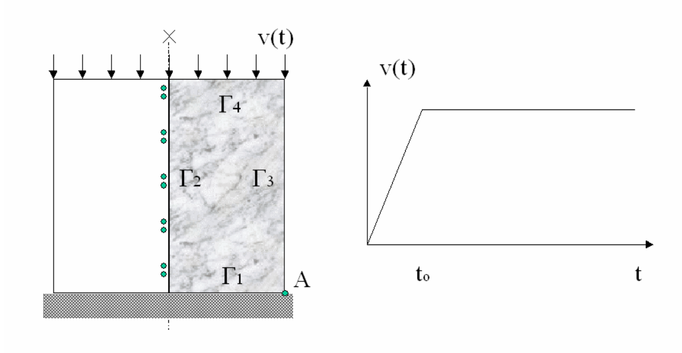
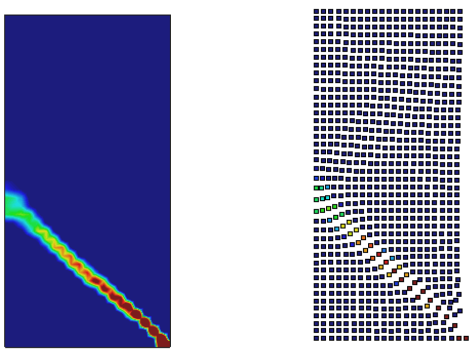


Figure 4: Sketch of the problem



(a) (b)

Figure 5: TSPH: (a) Viscoplastic strain; (b) Deformed configuration

5. CONCLUSION

The Taylor-SPH method applied to shock waves propagating in viscoplastic media has been presented. The method consists of a two-steps time discretization scheme based on a Taylor series expansion of the stress and velocity fields using a corrected SPH for the spatial discretization. Two different sets of particles have been used for the computations at each time step and a Lagrangian kernel has been used in order to avoid numerical instabilities. The results have been compared with those obtained using FEM. The method proposed herein is shown to be stable, robust and only a reduced number of particles is required to obtain reasonably accurate results.

6. REFERENCES

1. Belytschko T. , Guo Y., Liu W. K. and Xiao S. P., A unified stability analysis of meshless particle methods , *Int. J. Numer. Meth. Engrg.* 48 (2000) 1359-1400.
2. Belytschko T., Krongauz Y., Organ D., Fleming M., and Krysl P., Meshless methods: An overview and recent developments, *Comput. Methods Appl. Mech. Engrg* , 139 (1996) 3-47.
3. Benz W., Smooth particle hydrodynamics: a review, *The Numerical Modelling of Nonlinear Stellar Pulsatations*, J.R. Buchler (Ed), Kluwer Academic Publishers, (1990) 269-288.
4. Gingold R.A. & Monaghan J.J., Smoothed particles hydrodynamics: Theory and application to non-spherical stars, *Monthly Notices of the Royal Astronomical Society*, 181 (1977) 375-389.
5. Liu G.R. & Liu M.B., *Smoothed Particle Hydrodynamics: a meshfree particle method*, World Scientific (2003).
6. Lucy L.B., A numerical approach to the testing of fusion process, *Astronomical Journal*, 82 (1977) 1013-1024.
7. Mabssout M. & Pastor M., A Taylor–Galerkin algorithm for shock wave propagation and strain localization failure of viscoplastic continua, *Comput. Methods. Appl. Mech. Engrg*, 192 (2003) 955–971.
8. Mabssout M., Pastor M., Herreros M.I., Quecedo M., A Runge-Kutta, Taylor-Galerkin scheme for hyperbolic systems with source terms. Application to shock wave propagation in viscoplastic geomaterials, *Int. J. Numer. Analytical Meth. Geomechanics*, 30(13) (2006) 1337-1355.
9. Monaghan J.J. & Lattanzio J.C., A refined particle method for astrophysical problems, *Astronomy and Astrophysics*, 149 (1985) 135-143.
10. Perzyna P., Fundamental problems in viscoplasticity, Recent Advances in Applied Mechanics, *Academic press*, New York, 9 (1966) 243-377.