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Symmetry-invariant L.E.S. turbulence models for non-isothermal fluid flows

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Résumé
On présente le groupe de symétrie des équations de Navier–Stokes anisothermes. Ensuite, on construit une classe de modèles de turbulence qui conservent les propriétés physiques liées aux symétries des équations. Ces modèles sont testés dans le cas de la convection naturelle dans une cavité différentiellement chauffée.

Abstract
The symmetry group of the non-isothermal Navier–Stokes equations is presented. Then, a class of L.E.S. turbulence models which preserve the physical properties contained in the symmetry group is built. A model of the class is tested in the case of natural convection in a differentially heated cavity.

1 INTRODUCTION

Turbulence modeling constitutes an important challenge in engineering science and no turbulence model, neither with RANS method or with large-eddy simulation (LES) approach, is really satisfying regarding their mathematical and physical properties. As a consequence, Razafindralandy et al. showed in [13, 12] a way to build models preserving the physical properties of the flow using the theory of symmetry group [9]. The symmetry group of the governing equations seems to be the adequate tool which traduces mathematically the physical properties of the flow. Indeed, Noether's theorem [7, 9] says that, for a lagrangian system, each symmetry of the equation corresponds to a conservation law. For example, a system having a time-translation symmetry is energy conserving. In the field of fluid mechanics, the symmetry theory has been used to deduce scaling laws, such as algebraic, logarithmic (wall) or exponential laws [8], to study the dynamics of vorticies [4], or to investigate the spectral properties [14] of the flow

The work done in [12] also showed that the symmetry approach may give numerically satisfying models if the parameters are correctly set. However, this work concerns only isothermal flows. Though, as for isothermal flows, many turbulence models for non-isothermal flows are not compatible with the symmetry group of the governing equations [11]. In the present paper, we extend this work to the non-isothermal case.

This article is organized as follow. A reminder on the symmetry group theory is given in the next section. The Lie point symmetries of the Navier–Stokes equations for non-isothermal flows are presented. In section 3, a class of symmetry-preserving turbulence models is built. This class is refined in section 4 to get simpler models which can easily be integrated in a numerical schemes. One of these simple models is numerecally tested in section 5.

2 THE SYMMETRY GROUP OF THE EQUATIONS

Consider the Navier–Stokes equations for an incompressible, non-isothermal Newtonian fluid:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \dfrac{1}{\rho}\nabla p - \operatorname{div} \tau_r - \beta g \theta\, e_3 = 0 \\ \dfrac{\partial \theta}{\partial t} + \operatorname{div}(\theta \boldsymbol{u}) - \operatorname{div} \mathbf{h}_r = 0 \\ \operatorname{div} \boldsymbol{u} = 0 \end{cases} \tag{1}$$

In these expressions,

$$\tau_r = 2\nu \mathsf{S} \qquad \text{and} \qquad \mathbf{h}_r = \kappa \nabla \theta$$

are respectively the viscous stress tensor and the heat flux. S=$[\nabla \boldsymbol{u} + {}^T\nabla \boldsymbol{u}]/2$ is the strain rate tensor. β is the thermal expansion and g the gravity acceleration. To simplify, we also designate equations (1) by

$$E(\mathbf{q}) = 0 \qquad \text{with} \qquad \mathbf{q} = (t, \boldsymbol{x}, \boldsymbol{u}, p, \theta). \tag{2}$$

In this paper, we consider only one (or multi-)parameter transformations of the form

$$T_a \;:\; \mathbf{q} = (t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto \widehat{\mathbf{q}} = (\widehat{t}, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}}, \widehat{p}, \widehat{\theta}) \tag{3}$$

where $\widehat{\boldsymbol{q}} = \widehat{\boldsymbol{q}}(q, a)$ depends continuously on the real parameter a. We assume that the parametrization is such that a= 0 corresponds to the identity. Transformation (3) is a said a *symmetry* of (1) if, to each solution of (1), it corresponds another solution, i.e.

$$E(\mathbf{q}) = 0 \qquad \Longrightarrow \qquad E(\widehat{\mathbf{q}}) = 0. \tag{4}$$

The set of the one-parameter symmetries of (1) constitutes a (local) Lie group G called the *symmetry group* of (1). G is then characterized by its variation at the vicinity of a= 0 which is represented by the vector field or *infinitesimal generator*

$$X = \eta_t \frac{\partial}{\partial t} + \eta_x \frac{\partial}{\partial x} + \eta_y \frac{\partial}{\partial y} + \eta_z \frac{\partial}{\partial z} + \eta_u \frac{\partial}{\partial u} + \eta_v \frac{\partial}{\partial v} + \eta_w \frac{\partial}{\partial w} + \eta_p \frac{\partial}{\partial p} + \eta_\theta \frac{\partial}{\partial \theta} \tag{5}$$

Where

$$\eta_q = \left. \frac{\mathrm{d}\widehat{q}}{\mathrm{d}a} \right|_{a=0}. \tag{6}$$

Knowing the η_q 's, the expression of $\hat{\boldsymbol{q}}$ can be obtained by solving the system

$$\begin{cases} \dfrac{\mathrm{d}\widehat{\mathrm{q}}}{\mathrm{d}a} = \eta_q(\widehat{\mathrm{q}}), \\ \\ \widehat{\mathrm{q}}(\mathrm{q}, a=0) = \mathrm{q}. \end{cases} \tag{7}$$

For locally integrable equations, Lie's theory (see [9, 5]) permits to replace the symmetry condition (4) by the following one which provides an algorithmic way to compute all the symmetries of the equations:

$$\mathbf{X}^{(2)}.E = 0 \qquad \text{when} \qquad E(\mathbf{q}) = 0. \tag{8}$$

2

$\mathbf{X}^{(2)}$ is a prolongation of $\mathbf{X}$ which takes into account the derivative terms in E up to the second order. More precisely,

$$\begin{aligned} \mathbf{X}^{(2)} = \mathbf{X} &+ \eta_u^t \frac{\partial}{\partial u_t} + \eta_u^x \frac{\partial}{\partial u_x} + \eta_u^y \frac{\partial}{\partial u_y} + \eta_u^z \frac{\partial}{\partial u_z} + \eta_u^{xx} \frac{\partial}{\partial u_{xx}} + \eta_u^{yy} \frac{\partial}{\partial u_{yy}} + \eta_u^{zz} \frac{\partial}{\partial u_{zz}} \\ &+ \eta_v^t \frac{\partial}{\partial v_t} + \eta_v^x \frac{\partial}{\partial v_x} + \eta_v^y \frac{\partial}{\partial v_y} + \eta_v^z \frac{\partial}{\partial v_z} + \eta_v^{xx} \frac{\partial}{\partial v_{xx}} + \eta_v^{yy} \frac{\partial}{\partial v_{yy}} + \eta_v^{zz} \frac{\partial}{\partial v_{zz}} \\ &+ \eta_w^t \frac{\partial}{\partial w_t} + \eta_w^x \frac{\partial}{\partial w_x} + \eta_w^y \frac{\partial}{\partial w_y} + \eta_w^z \frac{\partial}{\partial w_z} + \eta_w^{xx} \frac{\partial}{\partial w_{xx}} + \eta_w^{yy} \frac{\partial}{\partial w_{yy}} + \eta_w^{zz} \frac{\partial}{\partial w_{zz}} \\ &+ \eta_p^t \frac{\partial}{\partial p_t} + \eta_p^x \frac{\partial}{\partial p_x} + \eta_p^y \frac{\partial}{\partial p_y} + \eta_p^z \frac{\partial}{\partial p_z} \\ &+ \eta_\theta^t \frac{\partial}{\partial \theta_t} + \eta_\theta^x \frac{\partial}{\partial \theta_x} + \eta_\theta^y \frac{\partial}{\partial \theta_y} + \eta_\theta^z \frac{\partial}{\partial \theta_z} + \eta_\theta^{xx} \frac{\partial}{\partial \theta_{xx}} + \eta_\theta^{yy} \frac{\partial}{\partial \theta_{yy}} + \eta_\theta^{zz} \frac{\partial}{\partial \theta_{zz}}. \end{aligned} \tag{9}$$

The coefficients are defined recursively by:

$$\eta_q^s = D_s \eta_q - \sum_{r=t,x,y,z} \frac{\partial q}{\partial r} D_r \eta_s. \tag{10}$$

D is the total derivation operator.

When condition (8) is applied, we get equations on the η_q's, which resolution gives the following infinitesimal generators of (1)

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \tag{11}$$

$$\mathbf{X}_2 = \zeta(t)\frac{\partial}{\partial p}, \tag{12}$$

$$\mathbf{X}_3 = \beta g\, x_3 \frac{\partial}{\partial p} + \frac{1}{\rho}\frac{\partial}{\partial \theta} \tag{13}$$

$$\mathbf{X}_4 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}, \tag{14}$$

$$\mathbf{X}_{4+i} = \alpha_i(t)\frac{\partial}{\partial x_i} + \dot{\alpha}_i(t)\frac{\partial}{\partial u_i} - \rho\, x_i \ddot{\alpha}_i(t)\frac{\partial}{\partial p} \qquad i = 1,2,3, \tag{15}$$

$$\mathbf{X}_8 = 2t\frac{\partial}{\partial t} + \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j} - \sum_{j=1}^{3} u_j \frac{\partial}{\partial u_j} - 2p\frac{\partial}{\partial p} - 3\theta\frac{\partial}{\partial \theta} \tag{16}$$

where ζ and α_i's are arbitrary functions of time. The dot symbol ($\dot{}$) stands for derivation.

We can also consider symmetries (which are sometimes called equivalence transformations) of the form

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta, \nu, \kappa) \longmapsto (\widehat{t}, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}}, \widehat{p}, \widehat{\theta}, \widehat{\nu}, \widehat{\kappa}). \tag{17}$$

Such symmetries take a solution of (1) into a solution of other non-isothermal Navier–Stokes equations with a different value of ν and κ. Applying condition (8) leads then to the infinitesimal generator

$$\mathbf{X}_9 = \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial u_j} + 2p\frac{\partial}{\partial p} + \theta\frac{\partial}{\partial \theta} + 2\nu\frac{\partial}{\partial \nu} + 2\kappa\frac{\partial}{\partial \kappa}. \tag{18}$$

With these generators, we deduce the symmetry groups of (1) using (7). They are:

- the group of *time translations*, corresponding to $\mathbf{X}_1$:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t + a, \boldsymbol{x}, \boldsymbol{u}, p, \theta), \tag{19}$$

- the group of *pressure translations*, corresponding to $\mathbf{X}_2$:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}, \boldsymbol{u}, p + \zeta(t), \theta), \tag{20}$$

- the group of *pressure-temperature translations*, corresponding to $\mathbf{X}_3$:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}, \boldsymbol{u}, p + a\,\beta g\, x_3, \theta + a\frac{1}{\rho}), \tag{21}$$

- the group of *horizontal rotations*, corresponding to $\mathbf{X}_4$:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \mathbb{R}\boldsymbol{x}, \mathbb{R}\boldsymbol{u}, p, \theta) \tag{22}$$

where $\mathbb{R}$ is a 2D (constant) rotation matrix, with $\mathbb{R}^T \mathbb{R} = I_d$ and $\det \mathbb{R} = 1$, I_d being the identity matrix

- the (three-parameter) group of *generalized Galilean transformations*, spanned by the $\mathbf{X}_{4+i}$'s, i=1,2,3:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}+\boldsymbol{\alpha}(t), \boldsymbol{u}+\dot{\boldsymbol{\alpha}}(t), p+\rho\, \boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t), \theta), \tag{23}$$

- the group of the *first scaling transformations* generated by $\mathbf{X}_8$:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (\mathrm{e}^{2a}\, t, \mathrm{e}^{a}\, \boldsymbol{x}, \mathrm{e}^{-a}\, \boldsymbol{u}, \mathrm{e}^{-2a}\, p, \mathrm{e}^{-3a}\, \theta) \tag{24}$$

which shows how $\boldsymbol{u}$, p and θ change when the spatio-temporal scale is multiplied by (e^{a}, e^{2a}),

- and the group of the *second scaling transformations* corresponding to $\mathbf{X}_9$:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta, \nu, \kappa) \longmapsto (t, \mathrm{e}^{a}\, \boldsymbol{x}, \mathrm{e}^{a}\, \boldsymbol{u}, \mathrm{e}^{2a}\, p, \mathrm{e}^{a}\, \theta, \mathrm{e}^{2a}\, \nu, \mathrm{e}^{2a}\, \kappa) \tag{25}$$

which shows the consequence of the modification of the spatial scale.

The space translations correspond to transformations (23) where $\boldsymbol{\alpha}=(\alpha_i)_i$ is constant and the classical Galilean transformation to the case where $\boldsymbol{\alpha}$ is linear.

Other known symmetries of the non-isothermal Navier–Stokes equations (1), which could not be computed like the previous ones, are

- the *reflections*:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \mapsto (t, \Lambda\boldsymbol{x}, \Lambda\boldsymbol{u}, p, \iota_3\theta) \tag{26}$$

$$\Lambda = \begin{pmatrix} \iota_1 & 0 & 0 \\ 0 & \iota_2 & 0 \\ 0 & 0 & \iota_3 \end{pmatrix} \qquad \text{with} \qquad \iota_i = \pm 1, \quad i = 1, 2, 3,$$

- and the *material indifference* in the limit of a 2D horizontal flow in a simply connected domain ([1]) which is a time-dependent rotation:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p) \mapsto (t, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}}, \widehat{p}), \tag{27}$$

with

$$\widehat{\boldsymbol{x}} = \mathsf{R}(t)\ \boldsymbol{x}, \qquad \widehat{\boldsymbol{u}} = \mathsf{R}(t)\ \boldsymbol{u} + \dot{\mathsf{R}}(t)\ \boldsymbol{x}, \qquad \widehat{p} = p - 3\omega\phi + \frac{1}{2}\omega^2 \|\boldsymbol{x}\|^2$$

where R(t) is an horizontal 2D rotation matrix with angle wt, w an arbitrary real constant, ϕ *the usual 2D stream function defined by:*

$$\boldsymbol{u} = \mathrm{curl}(\phi \boldsymbol{e}_3)$$

And ||•|| indicates the Euclidian norm.

The combination of all these symmetries constitutes a group, called the *symmetry group* of (1). In the next section, we build a class of subgrid-scale models which are compatible with the symmetry group.

3 SYMMETRY AND L.E.S. MODELING

Large-eddy simulation consists in computing the filtered fields ($\overline{\boldsymbol{u}}$,$\bar{p}$,$\bar{\theta}$) which are used as direct approximations of the actual velocity $\boldsymbol{u}$, pressure p and temperature θ of the fluid. For this, a turbulence model is introduced. A "good" turbulence model is one with which has the ($\overline{\boldsymbol{u}}$,$\bar{p}$,$\bar{\theta}$) same properties as $(\boldsymbol{u}, p, \theta)$ from certain point of view. In our approach, we require that $(\boldsymbol{u}, p, \theta)$ has the same symmetry properties as $(\boldsymbol{u}, p, \theta)$. This requirement is important because, as we announced earlier, the symmetry group contains important physical information on the flow.

The equations of $\overline{\boldsymbol{u}}$ are

$$\begin{cases} \dfrac{\partial \overline{\boldsymbol{u}}}{\partial t} + \operatorname{div}(\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + \dfrac{1}{\rho}\nabla \overline{p} - \operatorname{div}(\overline{\tau}_r - \tau) - \beta g \overline{\theta}\, \boldsymbol{e}_3 = 0 \\ \\ \dfrac{\partial \overline{\theta}}{\partial t} + \operatorname{div}(\overline{\theta}\overline{\boldsymbol{u}}) - \operatorname{div}(\overline{\mathbf{h}}_r - \mathbf{h}) = 0 \\ \\ \operatorname{div} \overline{\boldsymbol{u}} = 0. \end{cases} \tag{28}$$

The symbole bar ($^{-}$) stands for filtering. $\tau = \overline{u \otimes u} - \bar{u} \otimes \bar{u}$ is the subgrid stress tensor and $\mathbf{h}$ = $\overline{\theta u}$ - $\bar{\theta}$ $\bar{u}$ the subgrid flux.

τ and $\mathbf{h}$ must be modelled to close the equations. The model should be such that each symmetry of (1) apllied to ($\boldsymbol{u}, p, \theta$ is also a symmetry of (28) applied to ($\overline{\boldsymbol{u}}$,$\bar{p}$,$\bar{\theta}$) When it is the case, the model will be said *invariant*.

In [13, 11] some popular turbulence models have been analyzed regarding their invariance. For the non-isothermal case, the Smagorinsky model, the dynamic model [6], Eidson's model [3], the modified Eidson's model by Peng and Davidson [10], the similarity model and the mixed Smagorinsky-similarity model were considered. The result of this analysis is recalled on table 1. It can be seen on it that only few models fit the invariance condition under all the symmetries of the equations. The dynamic and the similarity models are invariant, but only if the test filter does not destroy the symmetries. This observation leads us to propose new models which are compatible with the symmetries of the equations.

Translations (19), (20), (21) and (23) remains symmetries of (28) if τ and $\mathbf{h}$ depend only on $\bar{S}$ and $\overline{\mathbb{T}}$ = $\nabla\bar{\theta}$:

$$-\tau = \mathfrak{L}(\overline{S}, \overline{\mathbb{T}}) \tag{29}$$

$$-\mathbf{h} = \mathfrak{F}(\overline{S}, \overline{\mathbb{T}}). \tag{30}$$

Next, rotations (22), reflexions (26) and the material indifference (27) are symmetries of (28) if the model has the following form

$$\begin{cases} -\tau^d = E_1\overline{S} + E_2(\operatorname{Adj}\overline{S})^d + E_3(\overline{\mathbb{T}}\otimes\overline{\mathbb{T}})^d \\ \qquad + E_4\left[\overline{S}(\overline{\mathbb{T}}\otimes\overline{\mathbb{T}})\right]^d + E_5\left[\overline{S}(\overline{\mathbb{T}}\otimes\overline{\mathbb{T}})\overline{S}\right]^d, \\ -\mathbf{h} = E_6\overline{\mathbb{T}} + E_7\overline{S}\,\overline{\mathbb{T}} + E_8\overline{S}^2\overline{\mathbb{T}}, \end{cases} \tag{31}$$

where the coefficients E_i are scalar functions of the invariants obtained from $\bar{S}$ and $\overline{\mathbb{T}}$ which are:

	Time, pressure, Galilean	**Pressure-temperature**	**Rotation, reflection, material**	**Scaling**
Smagorinsky	invariant	invariant	invariant	***non-invariant***
Dynamic	invariant	invariant	invariant	invariant
Eidson	invariant	invariant	invariant	***non-invariant***
Modified Eidson	invariant	invariant	invariant	***non-invariant***
Similarity	invariant	invariant	invariant	invariant
Mixed	invariant	invariant	invariant	***non-invariant***

Table 1: *Result of the model analysis. See Razafindralandy and Hamdouni [11].*

$$\mathcal{X} = \operatorname{tr}\overline{S}^2, \qquad \xi = \det\overline{S}, \qquad \vartheta = \overline{\mathbb{T}}^2, \qquad \omega_1 = \overline{\mathbb{T}}\,.\,\overline{S}\,\overline{\mathbb{T}}, \qquad \omega_2 = \overline{S}\,\overline{\mathbb{T}}\,.\,\overline{S}\,\overline{\mathbb{T}},$$

and Adj is the adjoint operator defined by

$$\overline{S}.(\operatorname{Adj}\overline{S}) = (\det\overline{S})\ I_d.$$

Expressions (31) is obtained from the theory of invariants and the equality of traces.

Equations (28) is invariant under the scale transformation (24) if and only if

$$\hat{\tau} = \mathrm{e}^{-2a}\,\tau \qquad \text{and} \qquad \widehat{\overline{\mathbb{T}}} = \mathrm{e}^{-4a}\,\overline{\mathbb{T}},$$

that is

$$E_i(\widehat{\mathcal{X}},\widehat{\xi},\widehat{\vartheta},\widehat{\omega}_1,\widehat{\omega}_2) = \mathrm{e}^{b_i a}\ E_i(\mathcal{X},\ \xi,\ \vartheta,\ \omega_1,\ \omega_2), \qquad i = 1,...,8 \tag{32}$$

with

$$b_1 = 0,\quad b_2 = 2,\quad b_3 = 6,\quad b_4 = 8,\quad b_5 = 10,\quad b_6 = 0,\quad b_7 = 2,\quad b_8 = 4.$$

Deriving with respect to a and taking $a = 0$ give:

$$\mathcal{X}\frac{\partial E_i}{\partial \mathcal{X}} + \frac{3}{2}\xi\frac{\partial E_i}{\partial \xi} + 2\vartheta\frac{\partial E_i}{\partial \vartheta} + \frac{5}{2}\omega_1\frac{\partial E_i}{\partial \omega_1} + 3\omega_2\frac{\partial E_i}{\partial \omega_2} = c_i E_i \tag{33}$$

where

$$c_1 = 0,\ \ c_2 = -\frac{1}{2},\ \ c_3 = -\frac{3}{2},\ \ c_4 = -2,\ \ c_5 = -\frac{5}{2},\ \ c_6 = 0,\ \ c_7 = -\frac{1}{2},\ \ c_8 = -1.$$

The associated characteristic equations are

$$\frac{\mathrm{d}\,\mathcal{X}}{\mathcal{X}} = \frac{2}{3}\frac{\mathrm{d}\,\xi}{\xi} = \frac{\mathrm{d}\,\vartheta}{2\vartheta} = \frac{2}{5}\frac{\mathrm{d}\,\omega_1}{\omega_1} = \frac{\mathrm{d}\,\omega_2}{3\omega_2} = \frac{\mathrm{d}\,E_i}{c_i E_i}. \tag{34}$$

Hence, the E_i must be of the form:

$$E_i(\mathcal{X},\ \xi,\ \vartheta,\ \omega_1,\ \omega_2) = \mathcal{X}^{c_i} E_i'(v_1, v_2, v_3, v_4)$$

where the v_i's are the invariants:

$$v_1 = \frac{\xi}{\mathcal{X}^{3/2}},\quad v_2 = \frac{\vartheta}{\mathcal{X}^2},\quad v_3 = \frac{\omega_1}{\mathcal{X}^{5/2}},\quad v_4 = \frac{\omega_2}{\mathcal{X}^3}. \tag{35}$$

Finally, the second scaling transformation (25) is a symmetry of equations (28) if

$$E_i' = \nu F_i, \qquad i = 1,...,5$$

And

$$E_i' = \kappa F_i, \qquad i = 6,...,8.$$

This condition is only sufficient, not necessary.

To sum up, we get the following class of subgrid models which are consistent with the symmetry group of (1):

$$\begin{cases} -\tau^d = \nu F_1 \overline{S} + \nu \mathcal{X}^{-1/2} F_2 \,\mathrm{Adj}^d\, \overline{S} \\ \qquad + \nu \mathcal{X}^{-3/2} F_3 (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d + \nu \mathcal{X}^{-2} F_4 [\overline{S}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})]^d] \\ \qquad + \nu \mathcal{X}^{-5/2} F_5 \overline{S}[(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})\overline{S}]^d] \\ -\mathbf{h} = \kappa \left(F_6 + \mathcal{X}^{-1/2} F_7 \overline{S} + \mathcal{X}^{-1} F_8 \overline{S}^2 \right) \overline{\mathbb{T}}, \end{cases} \tag{36}$$

This class of models contains eight arbitrary functions. This number can be lowered using some hypothesis that we shall see in the next section.

4 MODEL SIMPLIFICATION

In order to reduce the degree of freedom of the model, we propose to restrain the class (36) to models which derive from a potential. This restriction is legitimated by the fact that, like τ_r and $\mathbf{h}_r$, τ and $\mathbf{h}$ represent respectively a (subgrid) stress and a (subgrid) heat flux. Moreover, τ_r and $\mathbf{h}_r$ derive from the potentials $\nu\, tr\bar{S}^2$ and $\kappa\|\overline{\mathbb{T}}\|^2/2$, in the sense that

$$\tau_r = \frac{\partial \nu \ \mathrm{tr}\overline{\mathrm{S}}^2}{\partial \overline{\mathrm{S}}} \qquad \text{and} \qquad \mathbf{h}_r = \frac{\partial}{\partial \mathbb{T}} \left(\frac{1}{2} \kappa ||\overline{\mathbb{T}}||^2 \right) \tag{37}$$

Hence, τ and $\mathbf{h}$ should also derive from potentials. This condition leads to the following class of models (see [13]):

$$\begin{cases} -\tau^d = \nu \left[2g_m - 3v_1 \dfrac{\partial g_m}{\partial v_1} - 4v_2 \dfrac{\partial g_m}{\partial v_2} - 5v_3 \dfrac{\partial g_m}{\partial v_3} - 6v_4 \dfrac{\partial g_m}{\partial v_4} \right] \overline{S} \\ \qquad + \nu \left[\mathcal{X}^{-1/2} \dfrac{\partial g_m}{\partial v_1} \mathrm{Adj}^d\, \overline{S} + \mathcal{X}^{-3/2} \dfrac{\partial g_m}{\partial v_3} (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d + 2\mathcal{X}^{-2} \dfrac{\partial g_m}{\partial v_4} [\overline{S}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})]^d \right], \\ -\mathbf{h} = \kappa \left(\dfrac{\partial g_t}{\partial v_2} I_3 + \mathcal{X}^{-1/2} \dfrac{\partial g_t}{\partial v_3} \overline{S} + \mathcal{X}^{-1} \dfrac{\partial g_t}{\partial v_4} \overline{S}^2 \right) \overline{\mathbb{T}}, \end{cases} \tag{38}$$

where g_m and g_t are arbitrary functions of the v_i's

Note that the assumption that the model derives from a convex potential ensure the stability of the model [13].

Further simplifications can be done on the class (38) according to the type of model we wish to get.

4.1 Strongly coupled model

If g_m and g_t are only functions of v_1 and v_2, we get:

$$\begin{cases} -\tau^d = \nu\left(2g_m - 3v_1\dfrac{\partial g_m}{\partial v_1} - 4v_2\dfrac{\partial g_m}{\partial v_2}\right)\overline{S} + \nu\dfrac{1}{\|\overline{S}\|}\dfrac{\partial g_m}{\partial v_1}\mathrm{Adj}^d\overline{S}, \\ -\mathbf{h} = \kappa h_t \overline{\mathbb{T}}. \end{cases} \tag{39}$$

where $h_t = \frac{\partial g_t}{\partial v_2}$. This is a strongly coupled model in the sense that τ and **h** depend on both $\bar{S}$ and $\bar{\theta}$.

4.2 Decoupled model

We can obtain a decoupled model where τ does not depend on the temperature. For this, we can take g_m function only of v_1 and g_t function of v_2. It follows:

$$\begin{cases} -\tau^d = \nu(2g_m - 3v\dot{g}_m)\overline{S} + \nu\dfrac{1}{\|\overline{S}\|}\,\dot{g}_m\mathrm{Adj}^d\overline{S}, \\ -\mathbf{h} = \kappa h_t\,\overline{\mathbb{T}}, \end{cases} \tag{40}$$

4.3 Linear model

Let g_m and h_t be linear functions of v:

$$g_m = C_m v, \quad h_t = C_t v$$

where C_m are C_t are the constants of the model, which may depend on the grid size. In this case,

$$\begin{cases} -\tau^d = \nu C_m\left(-\det\overline{S}\dfrac{1}{\|\overline{S}\|^3}\overline{S} + \mathrm{Adj}^d\,\overline{S}\dfrac{1}{\|\overline{S}\|}\right), \\ -\mathbf{h} = \kappa C_t\dfrac{\det\overline{S}}{\|\overline{S}\|^3}\,\overline{\mathbb{T}}, \end{cases} \tag{41}$$

In the next section, we carry out a numerical test with the linear model (41) in the case of natural convection.

5 NUMERICAL TEST

We consider an air flow in a differentially heated cavity (see figure 1). The thermal expansion and the Prandtl number are such that

$$\beta g = 0.03, \text{ and } P_r = 0.711.$$

The code used for the simulation is based on a finite difference scheme [2], explicit in time. The time step is 2×10^{-4} s and the grid size (62 ×62×18). We use the experimental results in [15] as reference solution.

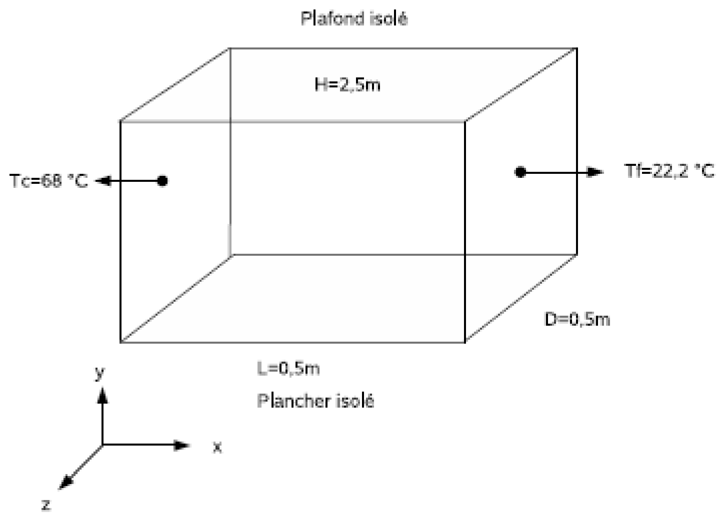

Figure 1: Differentially heated cavity.

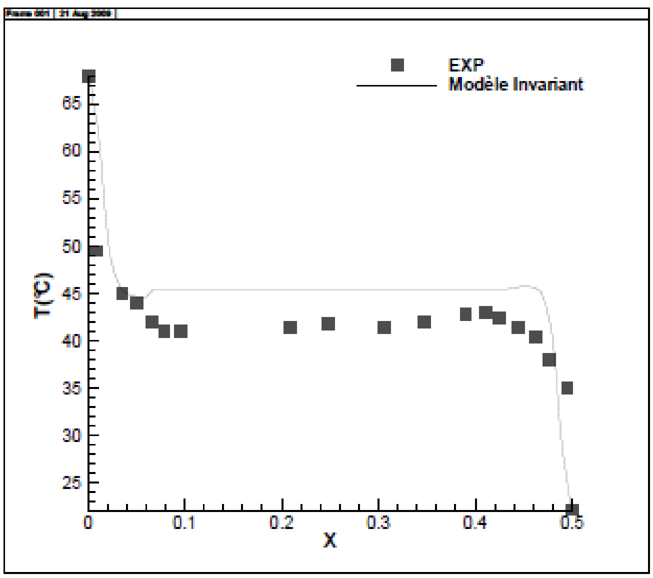

Figure 2: Mean temperature profile at y=1.25 m.

The temperature profile along an horizontal line, passing through the center of the cavity, is shown on figure (2). It can be observed that the model can give result close to the experimental data. This is more evident on the velocity profile which fits the experimental results as seen on figure (3). Note that these results were obtained with one of the simplest (linear) form of g_t and h_t.

6 CONCLUSION

As in the isothermal case, we showed that the symmetry approach may lead to numerically efficient models. Some improvements must however be done in the choice of the arbitrary functions involved in the model. But beyond the numerical results, we presented the possibility to build turbulence models which are compatible with the physical properties contained in the symmetry group of the equations.

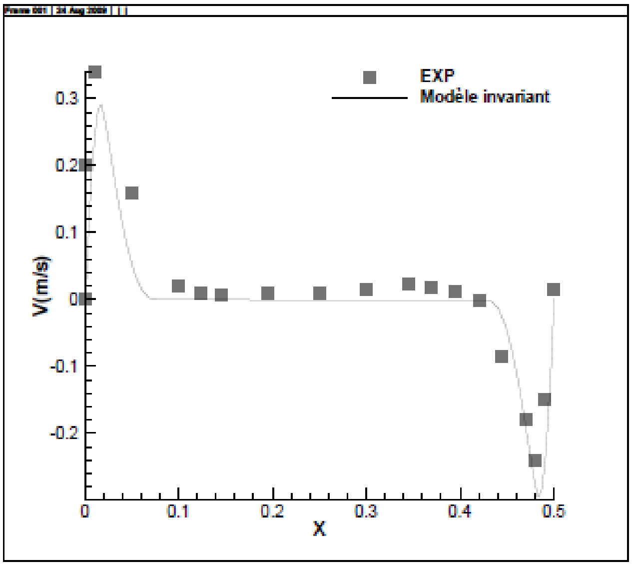

Figure 3: Mean velocity profile at y=1.25 m.

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