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Mathematical modling of subharmonic and superharmonic resonances of piezoelectric sandwich beams

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Résumé
En utilisant les structures à actionneurs et capteurs piézo-électriques et en tenant compte des non linéarités géométriques, un modèle de contrôle de vibrations non linéaires à été obtenu via une commande par rétroaction (feedback) proportionnelle et dérivée du potentiel électrique. Une méthodologie mathématique, basée sur la méthode des échelles multiples pour le contrôle des vibrations non linéaires aux résonances secondaires a été élaborée. Le contrôle actif des résonances sous harmoniques et sur harmoniques a été développé. Des modèles mathématiques fréquence-amplitude, phase-amplitude et la réponse temporelle tenant compte des paramètres de rétroaction ont été obtenus. Les effets des paramètres de feedback ont été analysés pour les petites et grandes amplitudes de vibrations des poutres sandwich et ces effets sur la suppression, l'amplification et la transition des résonnances sous et surharmonique ont été étudiés.

Abstract
The proportional and derivative nonlinear potential feedback controls via piezoelectric sensor and actuator layers are used. A simple and consistent model combining the geometrical nonlinear affects and the feedback control of a piezoelectric/elastic/piezoelectric sandwich beams is derived. A mathematical methodology based on the multiple scales method for nonlinear vibration control of secondary resonances is elaborated. The control of sub and superharmonic resonances is developed. The effects of the two feedback control parameters on the piezoelectric sandwich beams subjected to a harmonic excitation are studied for small and large vibration amplitudes. Frequency-amplitude, phase-amplitude and time response relationships related to sub and superharmonic resonances are explicitly given. The proportional and derivative feedback effects leading to suppression, amplification and translation of the sub and superharmonic resonances are investigated.
Keywords: piezoelectric, sandwich, nonlinear, multiple scales method, subharmonic, superharmonics resonance, sensor, actuator, active control, vibrations, instability, bifurcation.

1. INTRODUCTION

For slender structural elements, the nonlinear effects and particular by the geometric nonlinearities are of big importance in their dynamical behaviours at large amplitudes. It is well known that the vibration, when introduced in the structure, can grow up to large amplitudes and particularly for lightly damped structures. It is then necessary to control and suppress these undesired vibrations. One of the most popular approaches is the active vibration control with piezoelectric actuators and sensors. Piezoelectric material can be very well embedded in or attached on the lightweight structures, efficiently transform mechanical energy into electrical energy and vice versa and then give rise to active or adaptive structures. Subjected to dynamic loads, these structures may display a wealth of nonlinear phenomena including large amplitude oscillations in resonant regions, dynamic jumps, subharmonic and superharmonic oscillations, limit cycles, chaotic vibrations etc. This led to extensive research in active and passive vibration controls, and then to the development of a variety of mathematical models and methodological approaches for dynamical systems with nonlinear terms. Mathematical modeling of active feedback control of piezoelectric sandwich beams based on one harmonic balance method has been recently developed by Belouettar et al [1] and on the multiple scales method by Rechdaoui et al [2] for primary resonances. When the excitation amplitude is hard and its frequency is away from the linear natural frequency, the dynamical behaviour of the beam may be driven to new regimes called subharmonic and superharmonic oscillations [3].

Substantial researches have been carried out on beams and plates vibration suppression by active control using smart materials [1-10]. When a viscoelastic material damps the structure, the loss-factor is also amplitude dependent. Sandwich beams and plates with a viscoelastic core have been investigated by Daya et al. [11]. Simplified amplitude-frequency and amplitude loss-factor relationships are formulated. These papers presented a mathematical modeling to passively damp beams and to easily analyze their behaviors at large amplitudes. Based on a linear approximation of the electric potential through the thickness, a modal approach estimating the damping properties of sandwich beams has been presented by Duigou and al. [12]. With regard to the active control of nonlinear vibration of piezoelectric layers or patches, the existing reported work is very limited.

The present paper deals with the mathematical modeling and the numerical investigation of subharmonic and superharmonic resonances of sandwich piezoelectric beams under hard excitation amplitudes. A particular multiple scales method is developed in order to correctly account for the control parameters and the nonlinear coefficients of the governing equation. The frequency-amplitude and phase-amplitude relationships as well as the time response are explicitly given. Based on these analytical formulations, the proportional and derivative feedback parameters can be used to actively control the sub and superharmonic resonances.

2. MATHEMATICAL MODELLING

2.1 Equation of motion

A simple and consistent model combining the geometrical nonlinear effect and the feedback control of a piezoelectric/elastic/piezoelectric sandwich beams (figure 1) has been derived in [1]. The nonlinear effect produced by the large transverse vibration amplitude is modelled by a nonlinear strain-displacement relationship of von Karman type. The proportional and derivative potential feedback controls via sensor and actuator layers are used. More precisely, the actuator's potential $\varphi_A(x)$ is assumed to depend on the sensor's potential $\varphi_S(x)$ by the following control law:

$$\varphi_A = G_p \varphi_S + G_d \dot{\varphi}_S ; \qquad \varphi_s = \frac{e^*_{31} h_s}{\in^*_{31}} \left(u_{,x} + \frac{1}{2} w^2_{,x} - z_s w_{,xx}\right) \tag{1}$$

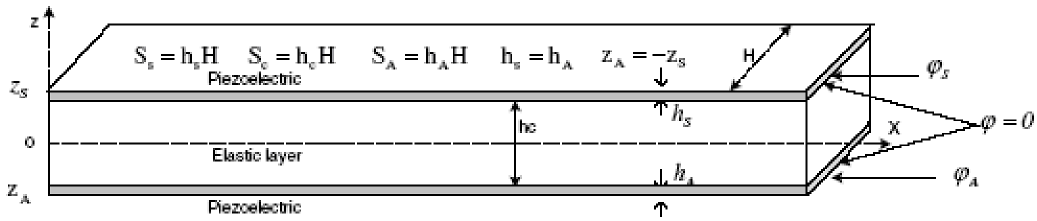

Figure. 1: Piezoelectric-elastic-piezoelectric beam

	elastic beam	piezoelectric layer
Length and width	1 m , H=5h	1 m , H=5h
Total thickness h=0.01m	$h_c = 5/6\,h$	$h_s = h_a = 1/12\,h$
Young's modulus	E_c = 6.9 10^{10} Pa	c^*_{11}=6.98 1010 Pa , $e^*_{31} = -23.2\,\frac{C}{m^2}$, $\in^*_{33}$=1.73 10^{-8} $\frac{F}{m}$
Mass density	$\rho_C = 2766\,\frac{kg}{m^3}$	$\rho_S = 7500\,\frac{kg}{m^3}$,

Table 1: Geometrical and material properties of the elastic beam and of the piezoelectric layer

where G_p and G_d are the gain proportional and derivative potentials of the considered feedback control respectively, u and w are the axial and transverse beam displacements. $e^*{}_{31}$, $\in^*{}_{31}$, h_s and z_s are piezo layer parameters, given in table 1. The structure is assumed to be transversally excited by the harmonic force $F_Z(x,t) = f(x)\cos(\omega t)$. Neglecting the axial displacement inertia, the equation of motion of the beam is reduced to the following nonlinear partial differential equation on the transverse displacement only [1]:

$$(\rho S)_* \ddot{w} + (EI)_* w_{,xxxx} - N(t)\, w_{,xx} - B_M\,(w^2_{,xx} + w_{,x} w_{,xxx}) \\ -(ES)_{pe} G_d z_s\,(\dot{w}_{,x} w_{,xxx} + 2\dot{w}_{,xx} w_{,xx} + w_{,x}\dot{w}_{,xxx} - z_s \dot{w}_{,xxxx}) = Fz \tag{2}$$

in which N(t) is the resultant axial force given by:

$$N(t) = \frac{1}{2L}(ES)_* \int_0^L w_{,x}{}^2\,dx - \frac{B_N}{L}\int_0^L w_{,xx}\,dx - \frac{(ES)_{pe}}{L} G_d \int_0^L (w_{,x}\dot{w}_{,x} - \dot{w}_{,xx} z_S)\,dx \tag{3}$$

$$(ES)_* = E_C S_C + 2c^*_{11} S_S + (ES)_{pe}(1-G_p) \quad ; \quad (ES)_{pe} = S_S \frac{(e^*_{31})^2}{\in^*_{33}} \quad ; \quad B_N = (ES)_{pe}(1-G_p) z_S \tag{4}$$

$$B_M = (ES)_{pe}(1+G_p) z_S \quad ; \quad (EI)_* = E_C I_C + 2c^*_{11}\left(I_S + S_S z_S^2\right) + \frac{(ES)_{pe}}{S_S}\left(2I_S + (1+G_p) z_S^2 S_S\right)$$

where the parameters $(ES)_*$ and $(EI)_*$ are the resulting extensional and bending stiffness of the piezo-sandwich beam, B_M and B_N are the rigidity terms and B_N displays coupling between the transverse bending and the axial stretching. In order to analyze the nonlinear vibration control of the piezoelectric/elastic/piezoelectric sandwich beams in a simplified way only one mode is considered $w(x,t) = q(t)\varphi_1(x)$ where $\varphi_1(x)$ is the free vibration mode of the sandwich beam and $q(t)$ is the associated time dependent amplitude. Based on the one-mode assumption, equation (2) is reduced to the following nonlinear differential equation [1]:

$$\ddot{q}(t)+2\mu\dot{q}(t)+\omega_L{}^2 q(t)+\alpha_2 q^2(t)+\alpha_3 q^3(t)+\alpha_4 q(t)\dot{q}(t)+\alpha_5 q^2(t)\dot{q}(t)=F_1\cos(\omega t) \qquad \text{(5-a)}$$

$$2\mu=\frac{(ES)_{pe}G_d z_S{}^2}{M}\int_0^L \varphi_{1,xxxx}(x)\varphi_1(x)dx \; ; \; \omega_L{}^2=\frac{(EI)_*}{M}\int_0^L \varphi_{1,xxxx}(x)\varphi_1(x)dx \; ;$$

$$M=(\rho S)_*\int_0^L (\varphi_1(x))^2 dx$$

$$\alpha_2=\frac{B_N}{M\,L}\int_0^L \varphi_{1,xx}(x)\varphi_1(x)dx\int_0^L \varphi_{1,xx}(x)dx-\frac{B_M}{M}\int_0^L\left\{(\varphi_{1,xx}(x))^2+\varphi_{1,x}(x)\varphi_{1,xxx}(x)\right\}\varphi_1(x)dx$$

$$\alpha_3=-\frac{(ES)_*}{2ML}\int_0^L \varphi_{1,xx}(x)\varphi_1(x)dx\int_0^L(\varphi_{1,x}(x))^2 dx$$

$$\alpha_4=-\frac{(ES)_{pe}G_d z_S}{M}\left(\frac{1}{L}\int_0^L \varphi_{1,xx}(x)\varphi_1(x)dx\int_0^L \varphi_{1,xx}(x)dx+2\left(\int_0^L\left\{(\varphi_{1,xx}(x))^2+\varphi_{1,x}(x)\varphi_{1,xxx}(x)\right\}\varphi_1(x)dx\right)\right)$$

$$\alpha_5=\frac{(ES)_{pe}G_d}{ML}\int_0^L \varphi_{1,xx}(x)\varphi_1(x)dx\int_0^L(\varphi_{1,x}(x))^2 dx; \qquad F_1=\frac{1}{M}\int_0^L f(x)\varphi_1(x)dx\,, \qquad \text{(5-b)}$$

Let us note that the coefficients of equation (5-a) depend on the control parameters G_p and G_d and these coefficients may be deeply influenced by the considered control law.

2.2 Method of solution

An approximate solution of equation (5-a), represented by an expansion in terms of different time scales, based on the multiple scales method, has been developed for small excitation in the vicinity of primary resonances [2]. When the excitation frequency is away from the linear natural frequency of the beam, and its amplitude is hard, the system may be driven to a new nonlinear regime by the so-called secondary resonances. The classical multiple scales method has been used for secondary resonances in [3]. It has to be noted that even if this classical procedure leads to simple and useful relationships, the effects of the quadratic terms α_2 and α_4 are disregarded in the resulting analytical relationships. To overcome this drawback, a new multiple scales strategy is developed in this paper.
For the analysis of the effects of large damping, quadratic and cubic nonlinearities, the damping, the nonlinearity and the excitation are needed to be ordered separately in the perturbation scheme. For this goal, the following perturbation equation is adopted:

$$\ddot{q}(t)+\omega_L{}^2 q(t)=-2\,\varepsilon\mu\,\dot{q}(t)-\varepsilon\,\alpha_2\,q^2(t)-\varepsilon^2\alpha_3 q^3(t)-\varepsilon\,\alpha_4\,q(t)\,\dot{q}(t)-\varepsilon^2\alpha_5 q^2(t)\dot{q}(t)+F_1\cos(\omega t) \qquad (6)$$

where ε is a small parameter. The second-order uniformly valued approximation of the solution of Eq. (6) is developed as :

$$q(t,\varepsilon)=q_0(T_0,T_1,T_2)+\varepsilon q_1(T_0,T_1,T_2)+\varepsilon^2 q_2(T_0,T_1,T_2)+\cdots \qquad (7)$$

A fast time scale $T_0=t$ and a slow time scales $T_1=\varepsilon t$ and $T_2=\varepsilon^2 t$, are introduced and the following time derivative are used: [13]

$$\frac{d}{dt}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2}+\cdots=D_0+\varepsilon D_1+\varepsilon^2 D_2+\cdots \qquad (8)$$

Remember that T_0 , T_1 and T_2 are new independent variables. Substituting Eqs. (7) and (8) into Eq.(6) and equating the coefficients of like powers of ε on both sides, the following differential system is obtained :

$$\begin{cases} D_0^{\,2}q_0+\omega_L^{\,2}q_0=F_1\cos(\omega t) & \text{(9-a)} \\ D_0^{\,2}q_1+\omega_L^{\,2}q_1=-2D_0D_1q_0-2\mu D_0q_0-\alpha_2 q_0^{\,2}-\alpha_4 q_0D_0q_0 & \text{(9-b)} \\ D_0^{\,2}q_2+\omega_L^{\,2}q_2=-2D_0D_1q_1-D_1^2q_0-2D_0D_2q_0-2\mu D_0q_1-2\mu D_1q_0-2\alpha_2q_1q_0 & \\ \qquad -\alpha_3q_0^{\,3}-\alpha_4q_0D_0q_1-\alpha_4q_0D_1q_0-\alpha_4q_1D_0q_0-\alpha_5q_0^{\,2}D_0q_0+F_1\cos(\omega T_1) & \text{(9-c)} \\ \ldots\ldots \end{cases}$$

The general solution of Eq. (9-a) in the complex form is given by:

$$q_0(T_0,T_1,T_2)=A(T_1,T_2)\,e^{i\omega_LT_0}+\Lambda e^{i\omega T_0}+cc \qquad ; \quad \Lambda=F_1/2(\omega_L^{\,2}-\omega^2) \tag{10}$$

where the cc stands for complex conjugate terms and $A(T_1,T_2)$ is the amplitude complex-valued quantity, function of the slow scales T_1 and T_2 that will be determined by imposing the solvability condition at the next level of the approximation. Substituting q_0 into (9-b) the following differential equation is obtained:

$$D_0^{\,2}q_1+\omega_L^{\,2}q_1=-2i\omega_L\left(\frac{\partial A}{\partial T_1}+\mu A\right)e^{i\omega_LT_0}-\left(\alpha_2+i\omega_L\alpha_4\right)A^2e^{2i\omega_LT_0}-\left(\alpha_2+i\omega\alpha_4\right)\Lambda^2e^{2i\omega T_0}$$
$$-\left(2\alpha_2+i\alpha_4(\omega_L+\omega)\right)\Lambda Ae^{i(\omega_L+\omega)T_0}-\left(2\alpha_2+i\alpha_4(\omega-\omega_L)\right)\Lambda\bar{A}e^{i(\omega-\omega_L)T_0}-2i\mu\Lambda\omega e^{i\omega T_0}-\alpha_2\left(A\bar{A}+\Lambda^2\right)+cc \tag{11}$$

Based on equation (11), various secondary resonance types, associated to the considered hard excitation, may be analysed. In this paper we restrict our studies to the case of the superharmonic $\omega\approx\frac{1}{2}\omega_L$ and the subharmonic $\omega\approx 2\,\omega_L$. The proportional and derivative feedback effects leading to suppression, amplification and translation of the superharmonic and subharmonic resonances will be investigated.

2. 2. 1. Superharmonic resonances $\omega\approx\frac{1}{2}\omega_L$

In the vicinity of $\frac{1}{2}\omega_L$, a detuning parameter σ is introduced and the excitation frequency ω is assumed to be $2\omega=\omega_L+\varepsilon^2\sigma$ where $\sigma=\mathrm{O}(1)$. The elimination of secular terms from the particular solution of (11) leads for the following complex amplitude equation:

$$2i\omega_L\left(\frac{\partial A}{\partial T_1}+\mu A\right)+\left(2\alpha_2+i\omega_L\alpha_4\right)\frac{\Lambda^2}{2}e^{i\varepsilon^2\sigma T_0}=0 \tag{12}$$

The solution of equation (11) in this case is given by:

$$q_1(T_0,T_1,T_2)=\left(\frac{\alpha_2}{3\omega_L^{\,2}}+\frac{i\alpha_4}{3\omega_L}\right)A^2e^{2i\omega_LT_0}+\frac{\Lambda}{\omega(\omega+2\omega_L)}\left(2\alpha_2+i\alpha_4(\omega_L+\omega)\right)Ae^{i(\omega_L+\omega)T_0}$$
$$+\frac{\Lambda}{\omega(\omega-2\omega_L)}\left(2\alpha_2+i\alpha_4(\omega-\omega_L)\right)\bar{A}e^{i(\omega-\omega_L)T_0}+\frac{2i\mu\Lambda\omega}{\omega^2-\omega_L^{\,2}}e^{i\omega T_0}-\frac{\alpha_2}{\omega_L^{\,2}}\left(A\bar{A}+\Lambda^2\right)+cc \tag{13}$$

Substituting Eqs. (10) and (13) into Eq. (9-c) and eliminating secular terms, one obtains:

$$\frac{\partial^2A}{\partial T_1^2}+2\mu\frac{\partial A}{\partial T_1}+2i\omega_L\frac{\partial A}{\partial T_2}+\left(\chi_1+i\chi_2\right)A^2\bar{A}+2\left(\chi_3+i\chi_2\right)\Lambda^2A+\frac{4\mu\Lambda^2}{3\omega_L}\left(\omega_L\alpha_4-2i\alpha_2\right)e^{i\sigma T_2}=0 \tag{14}$$

where:

$$\chi_1 = 3\alpha_3 - \frac{10}{3}\frac{\alpha_2{}^2}{\omega_L{}^2} - \frac{\alpha_4{}^2}{3} \;; \qquad \chi_2 = \omega_L\alpha_5 - \frac{\alpha_2\alpha_4}{\omega_L}; \qquad \chi_3 = 3\alpha_3 - \frac{46}{15}\frac{\alpha_2{}^2}{\omega_L{}^2} - \frac{4}{15}\alpha_4{}^2 \tag{15}$$

Differentiating Eq. (12) with respect to T_1 and using (14, 15) one gets:

$$2i\omega_L\frac{\partial A}{\partial T_2} - \mu^2 A + \left(\chi_1 + i\chi_2\right)A^2\overline{A} + 2\left(\chi_3 + i\chi_2\right)\Lambda^2 A + \frac{13}{12\omega_L}\left(\omega_L\alpha_4 - 2i\alpha_2\right)\mu\Lambda^2 e^{i\sigma T_2} = 0 \tag{16}$$

Based on some mathematical manipulations and using Eqs. (8, 12, 16), the following differential equation is obtained:

$$\begin{aligned} &2i\omega_L\left(\frac{dA}{dt} + \varepsilon\mu A\right) - \varepsilon^2\mu^2 A + \varepsilon^2\left(\left(\chi_1 + i\chi_2\right)A^2\overline{A} + 2\left(\chi_3 + i\chi_2\right)\Lambda^2 A\right) \\ &\qquad + \left(2\varepsilon\alpha_2 + \varepsilon^2\frac{13\mu}{6}\alpha_4 + i\left(\varepsilon\omega_L\alpha_4 - \varepsilon^2\frac{13\mu}{3\omega_L}\alpha_2\right)\right)\frac{\Lambda^2}{2}e^{i\varepsilon^2\sigma t} = 0 \end{aligned} \tag{17}$$

The complex amplitude A is given in the modulus and phase form as:

$$A = \frac{1}{2}a(T_1, T_2)\, e^{i\beta(T_1,T_2)} \tag{18}$$

Substituting (18) in (17) and separating real and imaginary parts yield the following differential system governing the amplitude and the phase *(a , β)* of the response:

$$\begin{cases} \dfrac{da}{dt} = -\left(\varepsilon\mu + \varepsilon^2\dfrac{\chi_2\Lambda^2}{\omega_L}\right)a - \varepsilon^2\dfrac{\chi_2}{8\omega_L}a^3 - \left(2\varepsilon\alpha_2 + \varepsilon^2\dfrac{13\mu}{6}\alpha_4\right)\dfrac{\Lambda^2}{2\omega_L}\sin(\sigma T_2 - \beta) - \left(\varepsilon\omega_L\alpha_4 + \varepsilon^2\dfrac{13\mu}{3\omega_L}\alpha_2\right)\dfrac{\Lambda^2}{2\omega_L}\cos(\sigma T_2 - \beta) \\ a\dfrac{d\beta}{dt} = -\varepsilon^2\left(\dfrac{\mu^2}{2\omega_L} - \dfrac{\chi_3\Lambda^2}{\omega_L}\right)a + \varepsilon^2\dfrac{\chi_1}{8\omega_L}a^3 + \left(2\varepsilon\alpha_2 + \varepsilon^2\dfrac{13\mu}{6}\alpha_4\right)\dfrac{\Lambda^2}{2\omega_L}\cos(\sigma T_2 - \beta) - \left(\varepsilon\omega_L\alpha_4 + \varepsilon^2\dfrac{13\mu}{3\omega_L}\alpha_2\right)\dfrac{\Lambda^2}{2\omega_L}\sin(\sigma T_2 - \beta) \end{cases} \tag{19}$$

Using the transformation $\gamma = \sigma T_2 - \beta = \varepsilon^2\sigma t - \beta$, this system is rewritten as:

$$\begin{cases} \dfrac{da}{dt} = -\left(\varepsilon\mu + \varepsilon^2\dfrac{\chi_2\Lambda^2}{\omega_L}\right)a - \varepsilon^2\dfrac{\chi_2}{8\omega_L}a^3 - \left(2\varepsilon\alpha_2 + \varepsilon^2\dfrac{13\mu}{6}\alpha_4\right)\dfrac{\Lambda^2}{2\omega_L}\sin(\gamma) - \left(\varepsilon\omega_L\alpha_4 + \varepsilon^2\dfrac{13\mu}{3\omega_L}\alpha_2\right)\dfrac{\Lambda^2}{2\omega_L}\cos(\gamma) \\ a\dfrac{d\gamma}{dt} = \varepsilon^2\left(\sigma + \dfrac{\mu^2}{2\omega_L} - \dfrac{\chi_3\Lambda^2}{\omega_L}\right)a - \varepsilon^2\dfrac{\chi_1}{8\omega_L}a^3 - \left(2\varepsilon\alpha_2 + \varepsilon^2\dfrac{13\mu}{6}\alpha_4\right)\dfrac{\Lambda^2}{2\omega_L}\cos(\gamma) + \left(\varepsilon\omega_L\alpha_4 + \varepsilon^2\dfrac{13\mu}{3\omega_L}\alpha_2\right)\dfrac{\Lambda^2}{2\omega_L}\sin(\gamma) \end{cases} \tag{20}$$

The periodic solutions corresponding to the nontrivial fixed points of equations (20) for the superharmonic resonance $\omega \approx \frac{1}{2}\omega_L$ are obtained by the following algebraic system:

$$\begin{cases} \left(\varepsilon\mu + \varepsilon^2\dfrac{\chi_2\Lambda^2}{\omega_L}\right)a + \varepsilon^2\dfrac{\chi_2}{8\omega_L}a^3 = -\left(2\varepsilon\alpha_2 + \varepsilon^2\dfrac{13\mu}{6}\alpha_4\right)\dfrac{\Lambda^2}{2\omega_L}\sin(\gamma) - \left(\varepsilon\omega_L\alpha_4 + \varepsilon^2\dfrac{13\mu}{3\omega_L}\alpha_2\right)\dfrac{\Lambda^2}{2\omega_L}\cos(\gamma) \\ \varepsilon^2\left(\sigma + \dfrac{\mu^2}{2\omega_L} - \dfrac{\chi_3\Lambda^2}{\omega_L}\right)a - \varepsilon^2\dfrac{\chi_1}{8\omega_L}a^3 = \left(2\varepsilon\alpha_2 + \varepsilon^2\dfrac{13\mu}{6}\alpha_4\right)\dfrac{\Lambda^2}{2\omega_L}\cos(\gamma) - \left(\varepsilon\omega_L\alpha_4 + \varepsilon^2\dfrac{13\mu}{3\omega_L}\alpha_2\right)\dfrac{\Lambda^2}{2\omega_L}\sin(\gamma) \end{cases} \tag{21}$$

When the vibration is in the steady-state, the resulting frequency and phase relationships are:

$$\left(\left(\varepsilon\mu+\varepsilon^2\frac{\chi_2\Lambda^2}{\omega_L}\right)a+\varepsilon^2\frac{\chi_2}{8\omega_L}a^3\right)^2+\left(\varepsilon^2\left(\sigma+\frac{\mu^2}{2\omega_L}-\frac{\chi_3\Lambda^2}{\omega_L}\right)a-\varepsilon^2\frac{\chi_1}{8\omega_L}a^3\right)^2$$
$$=\left(\left(2\varepsilon\alpha_2+\varepsilon^2\frac{13\mu}{6}\alpha_4\right)\frac{\Lambda^2}{2\omega_L}\right)^2+\left(\left(\varepsilon\omega_L\alpha_4+\varepsilon^2\frac{13\mu}{3\omega_L}\alpha_2\right)\frac{\Lambda^2}{2\omega_L}\right)^2 \tag{22}$$

$$\tan(\gamma+\theta)=\frac{\varepsilon\mu+\varepsilon^2\frac{\chi_2\Lambda^2}{\omega_L}+\varepsilon^2\frac{\chi_2}{8\omega_L}a^2}{\varepsilon^2\left(\sigma+\frac{\mu^2}{2\omega_L}-\frac{\chi_3\Lambda^2}{\omega_L}-\frac{\chi_1}{8\omega_L}a^2\right)}\ ;\qquad \tan(\theta)=\frac{6\omega_L^2\alpha_4+\varepsilon 26\mu\alpha_2}{\omega_L\left(12\alpha_2+\varepsilon 13\mu\alpha_4\right)} \tag{23}$$

The corresponding approximate time solution is:

$$q(t,\varepsilon)=a\cos(2\omega t-\gamma)+2\Lambda\cos(\omega t)+\varepsilon\left[\frac{a^2}{2}\left(\frac{\alpha_2}{3\omega_L{}^2}\cos(2(2\omega t-\gamma))-\frac{\alpha_4}{3\omega_L}\sin(2(2\omega t-\gamma))\right)\right.$$
$$+\frac{\Lambda a}{\omega(\omega+2\omega_L)}\left(2\alpha_2\cos(3\omega t-\gamma)-\alpha_4(\omega_L+\omega)\sin(3\omega t-\gamma)\right)$$
$$\left.+\frac{\Lambda a}{\omega(\omega-2\omega_L)}\left(2\alpha_2\cos(\omega t-\gamma)-\alpha_4(\omega_L-\omega)\sin(\omega t-\gamma)\right)-\frac{4\mu\Lambda\omega}{\omega^2-\omega_L{}^2}\sin(\omega t)-\frac{2\alpha_2}{\omega_L{}^2}(\frac{a^2}{4}+\Lambda^2)\right] \tag{24}$$

in which $a(t)$ and $\gamma(t)$ are solutions of the differential system (20). Including more independent time scales $(T_2,T_3,\cdots)$ in the analysis, the procedure could be considered to yield approximations of even higher orders.

2. 2. 2. Subharmonic resonances $\omega\approx 2\omega_L$

When ω is near $2\omega_L$, the following detuning parameter is used: $\omega=2\omega_L+\varepsilon\sigma$. So, secular terms are eliminated by expressing $(\omega-\omega_L)T_0$ in terms of $\omega_L T_0$, i.e: $(\omega-\omega_L)T_0=\omega_L T_0+\sigma T_1$ and one obtains

$$2i\omega_L\left(\frac{\partial A}{\partial T_1}+\mu A\right)+\left(2\alpha_2+i\omega_L\alpha_4\right)\Lambda\bar{A}e^{i\sigma T_2}=0 \tag{25}$$

The solution of equation (9-b) is then given by:

$$q_1(T_0,T_1,T_2)=\left(\frac{\alpha_2}{3\omega_L{}^2}+\frac{i\alpha_4}{3\omega_L}\right)A^2e^{2i\omega_L T_0}+\frac{\Lambda}{\omega(\omega+2\omega_L)}\left(2\alpha_2+i\alpha_4(\omega_L+\omega)\right)Ae^{i(\omega_L+\omega)T_0}$$
$$+\frac{\Lambda}{4\omega^2-\omega_L^2}\left(\alpha_2+i\alpha_4\omega\right)e^{2i\omega T_0}+\frac{2i\mu\Lambda\omega}{\omega^2-\omega_L{}^2}e^{i\omega T_0}-\frac{\alpha_2}{\omega_L{}^2}\left(A\bar{A}+\Lambda^2\right)+cc \tag{26}$$

Substituting Eqs. (10) and (26) into Eq. (9-c) and eliminating secular terms, one gets:

$$\frac{\partial^2 A}{\partial T_1^2}+2\mu\frac{\partial A}{\partial T_1}+2i\omega_L\frac{\partial A}{\partial T_2}+\left(\chi_1+i\chi_2\right)A^2\bar{A}+2\left(\chi_4+i\chi_5\right)\Lambda^2 A$$
$$\left(\left(\frac{\partial\bar{A}}{\partial T_1}-\frac{4}{3}\mu\bar{A}\right)\alpha_4+i\frac{8}{3\omega_L}\alpha_2\mu\bar{A}\right)\Lambda e^{i\sigma T_2}=0 \tag{27}$$

where:

$$\chi_4=3\alpha_3-\frac{7}{4}\frac{\alpha_2{}^2}{\omega_L{}^2}-\frac{3}{16}\alpha_4{}^2\ ,\qquad \chi_5=\omega_L\alpha_5-\frac{\alpha_2\alpha_4}{2\omega_L} \tag{28}$$

The use of Eq. (25) leads to:

$$\frac{\partial^2 A}{\partial T_1^2}+\mu\frac{\partial A}{\partial T_1}=\frac{1}{2\omega_L}\left(-\alpha_4\omega_L+2i\alpha_2\right)\Lambda\frac{\partial\bar{A}}{\partial T_1}e^{i\sigma T_2} \tag{29}$$

Equation (27) can be then rewritten as:

$$2i\omega_L\frac{\partial A}{\partial T_2}-\mu^2 A+\left(\chi_1+i\chi_2\right)A^2\bar{A}+2\left(\tilde{\chi}_4+i\chi_2\right)\Lambda^2 A+\left(-\frac{7}{3}\alpha_4+i\frac{8}{3\omega_L}\alpha_2\right)\mu\bar{A}\Lambda e^{i\sigma T_2}=0 \tag{30}$$

In which:

$$\tilde{\chi}_4=3\alpha_3-\frac{5}{4}\frac{\alpha_2{}^2}{\omega_L^2}-\frac{5}{16}\alpha_4{}^2$$

Eqs. (25) and (30) are the first two terms in the multiple scales analysis of:

$$2i\omega_L\left(\frac{dA}{dt}+\varepsilon\mu A\right)-\varepsilon^2\mu^2 A+\varepsilon^2\left(\left(\chi_1+i\chi_2\right)A^2\bar{A}+2\left(\tilde{\chi}_4+i\chi_2\right)\Lambda^2 A\right)$$
$$+\left(2\varepsilon\alpha_2-\frac{7}{3}\varepsilon^2\mu\alpha_4+i\left(\varepsilon\omega_L\alpha_4+\frac{8}{3\omega_L}\varepsilon^2\mu\alpha_2\right)\right)\bar{A}\Lambda e^{i\sigma T_2}=0 \tag{31}$$

The time response in this approximation is given by:

$$q(t,\varepsilon)=a\cos(\frac{1}{2}(\omega t-\gamma))+2\Lambda\cos(\omega t)+\varepsilon\left[\frac{a^2}{2}\left(\frac{\alpha_2}{3\omega_L{}^2}\cos(2(2\omega t-\gamma))-\frac{\alpha_4}{3\omega_L}\sin(2(2\omega t-\gamma))\right)\right.$$
$$+\frac{\Lambda a}{\omega(\omega+2\omega_L)}\left(2\alpha_2\cos(3\omega t-\gamma)-\alpha_4(\omega_L+\omega)\sin(3\omega t-\gamma)\right)$$
$$\left.+\frac{\Lambda}{2\omega^2-\omega_L^2}\left(\alpha_2\cos(2\omega t)-\alpha_4\omega\sin(2\omega t)\right)-\frac{4\mu\Lambda\omega}{\omega^2-\omega_L{}^2}\sin(\omega t)-\frac{2\alpha_2}{\omega_L{}^2}(\frac{a^2}{4}+\Lambda^2)\right] \tag{32}$$

Using Eq. (18) the following differential equation is obtained from (31):

$$\begin{cases}\dfrac{da}{dt}=-\left(\varepsilon\mu+\varepsilon^2\dfrac{\chi_2\Lambda^2}{\omega_L}\right)a-\varepsilon^2\dfrac{\chi_2}{8\omega_L}a^3-\left(\varepsilon\alpha_2-\dfrac{7}{6}\varepsilon^2\mu\alpha_4\right)\dfrac{\Lambda}{\omega_L}a\sin(\sigma T_2-2\beta)-\left(\varepsilon\dfrac{1}{2}\omega_L\alpha_4+\dfrac{8}{6\omega_L}\varepsilon^2\mu\alpha_2\right)\dfrac{\Lambda}{\omega_L}a\cos(\sigma T_2-2\beta)\\ a\dfrac{d\beta}{dt}=-\varepsilon^2\left(\dfrac{\mu^2}{2\omega_L}-\dfrac{\tilde{\chi}_4\Lambda^2}{\omega_L}\right)a+\varepsilon^2\dfrac{\chi_1}{8\omega_L}a^3+\left(\varepsilon\alpha_2-\dfrac{7}{6}\varepsilon^2\mu\alpha_4\right)\dfrac{\Lambda}{\omega_L}a\cos(\sigma T_2-2\beta)-\left(\varepsilon\dfrac{1}{2}\omega_L\alpha_4+\dfrac{8}{6\omega_L}\varepsilon^2\mu\alpha_2\right)\dfrac{\Lambda}{\omega_L}a\sin(\sigma T_2-2\beta)\end{cases} \tag{33}$$

Using the transformation $\gamma=\sigma T_2-2\beta=\varepsilon^2\sigma t-2\beta$, this system is rewritten as:

$$\begin{cases} \dfrac{da}{dt} = -\left(\varepsilon\mu + \varepsilon^2 \dfrac{\chi_2 \Lambda^2}{\omega_L}\right) a - \varepsilon^2 \dfrac{\chi_2}{8\omega_L} a^3 - \left(\varepsilon\alpha_2 - \dfrac{7}{6}\varepsilon^2 \mu\alpha_4\right)\dfrac{\Lambda}{\omega_L} a\sin(\gamma) - \left(\varepsilon \dfrac{1}{2}\omega_L \alpha_4 + \dfrac{8}{6\omega_L}\varepsilon^2 \mu\alpha_2\right)\dfrac{\Lambda}{\omega_L} a\cos(\gamma) \\ a\dfrac{d\gamma}{dt} = \varepsilon^2\left(\sigma + \dfrac{\mu^2}{\omega_L} - \dfrac{2\tilde{\chi}_4 \Lambda^2}{\omega_L}\right) a - \varepsilon^2 \dfrac{\chi_1}{4\omega_L} a^3 - \left(\varepsilon 2\alpha_2 - \dfrac{7}{3}\varepsilon^2 \mu\alpha_4\right)\dfrac{\Lambda}{\omega_L} a\cos(\gamma) + \left(\varepsilon\omega_L \alpha_4 + \dfrac{8}{3\omega_L}\varepsilon^2 \mu\alpha_2\right)\dfrac{\Lambda}{\omega_L} a\sin(\gamma) \end{cases} \tag{34}$$

Based on the equilibrium solutions of (34) the frequency and phase amplitude relationships are given by:

$$\left(\varepsilon^2\sigma + \varepsilon^2 \frac{\mu^2}{\omega_L} - \varepsilon^2 \frac{\chi_1}{4\omega_L} a^2 - \varepsilon^2 \frac{2\tilde{\chi}_4 \Lambda^2}{\omega_L}\right)^2 + \left(\varepsilon 2\mu + \varepsilon^2 \frac{\chi_2}{4\omega_L} a^2 + \varepsilon^2 \frac{2\chi_2 \Lambda^2}{\omega_L}\right)^2 = \left(\left(\varepsilon 2\alpha_2 - \frac{7}{3}\varepsilon^2 \mu\alpha_4\right)\frac{\Lambda}{\omega_L}\right)^2 + \left(\left(\varepsilon\omega_L \alpha_4 + \frac{8}{3\omega_L}\varepsilon^2 \mu\alpha_2\right)\frac{\Lambda}{\omega_L}\right)^2 \tag{35}$$

$$\tan(\gamma+\theta) = \frac{\varepsilon 2\mu + \varepsilon^2 \dfrac{\chi_2}{4\omega_L} a^2 + \varepsilon^2 \dfrac{2\chi_2 \Lambda^2}{\omega_L}}{\varepsilon^2\sigma + \varepsilon^2 \dfrac{\mu^2}{\omega_L} - \varepsilon^2 \dfrac{\chi_1}{4\omega_L} a^2 - \varepsilon^2 \dfrac{2\tilde{\chi}_4 \Lambda^2}{\omega_L}}; \quad \tan(\theta) = \frac{3\omega_L^2 \alpha_4 + 8\varepsilon\mu\alpha_2}{\omega_L\left(6\alpha_2 - 7\varepsilon\mu\alpha_4\right)} \tag{36}$$

These mathematical models will be use for the investigation of the subharmonic and the superharmonic resonances behaviours of the considered active beam.

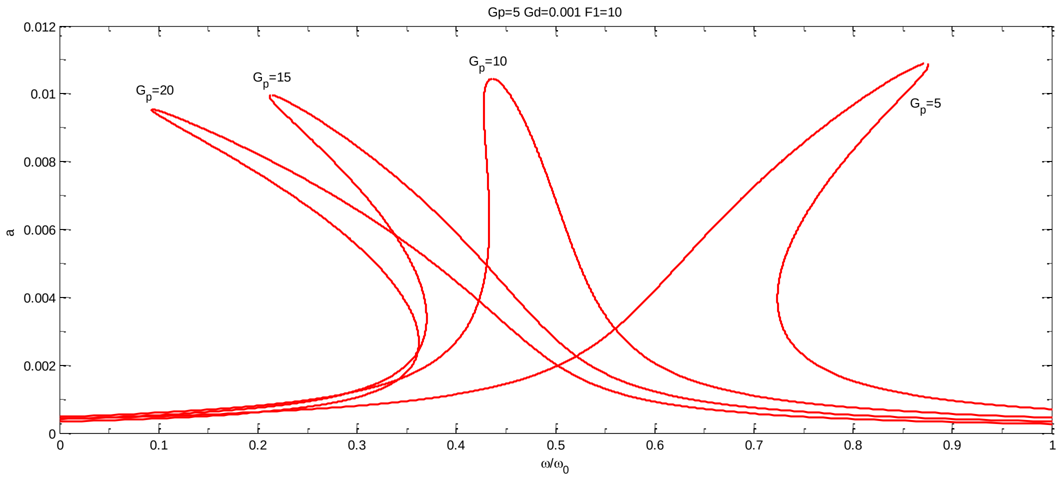

Figure. 2: Superharmonic nonlinear amplitude-frequency responses, $\omega \approx 1/2\,\omega_L$, for a uniformly excited sandwich S-S beam for various feedback parameters: G_p= 5, 10, 15, 20, G_d=0.01 and F_1=100.

3. NUMERICAL RESULTS

Numerical results are obtained for sandwich piezoelectric-elastic-piezoelectric beams as presented in figure 1. The material and geometrical properties of the host beam and of the piezoelectric layers are given in table 1. The simply supported boundary condition is considered in this analysis and the first eigenmode $\varphi_1(x) = \sin(\pi x / L)$ is used. For the superharmonic $\omega \approx \frac{1}{2}\omega_L$, the parameter Λ is approximated by $\Lambda \approx 2F_1/(3\omega_L{}^2)$. Figure. 2 presents the response curves and the effect of the proportional parameter G_p on the frequency-amplitude responses for the superharmonic resonance $\omega \approx \frac{1}{2}\omega_L$ for ϵ = 1. The transition from hard to soft nonlinear behaviours is clearly obtained.

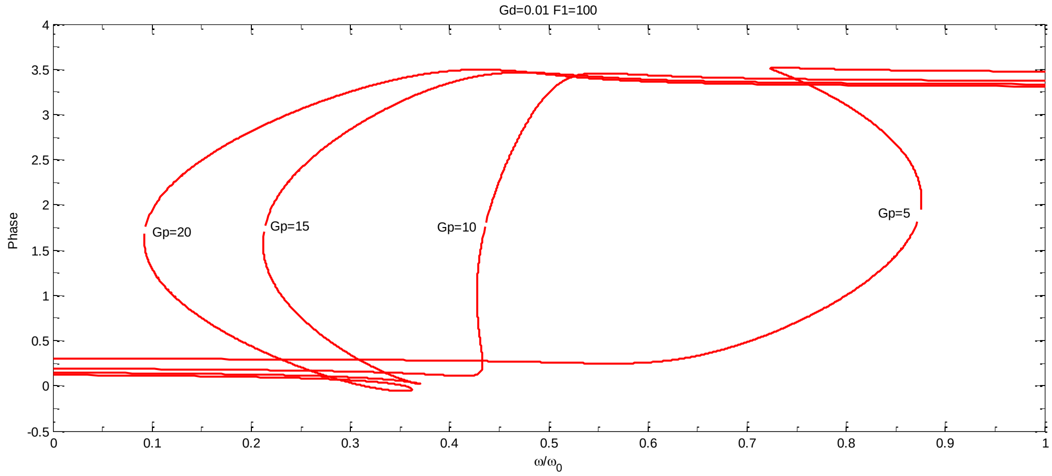

Figure. 3 Superharmonic nonlinear amplitude-frequency responses $\omega \approx 1/2\,\omega_L$ for a uniformly excited sandwich S-S beam for various feedback parameters: G_p=5, 10, 15, 20, G_d=0.01 and F_1=100.

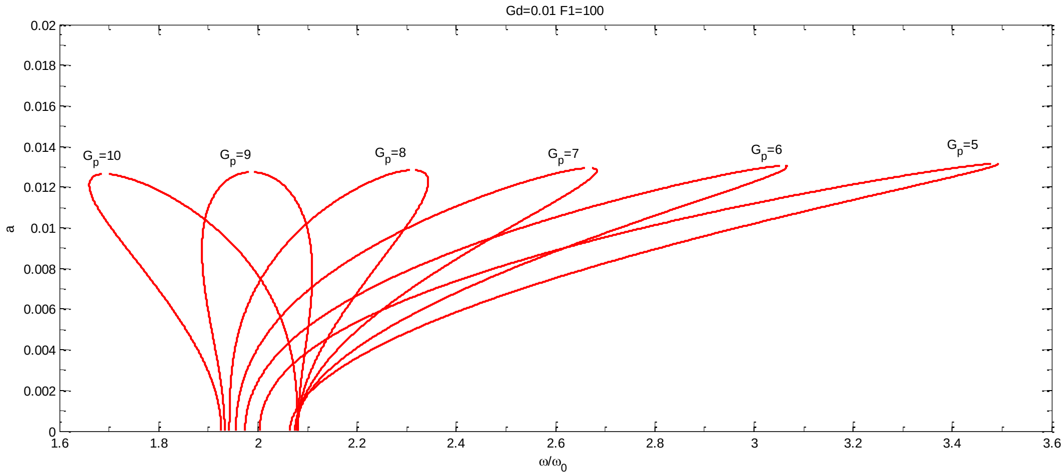

Figure. 4: Subharmonic nonlinear amplitude-frequency responses $\omega \approx 2\,\omega_L$ for a uniformly excited sandwich S-S beam for various feedback parameters: G_p=5, 6, 7, 8, 9, 10, G_d=0.01 and F_1=100.

For large amplitudes, the peak amplitude is decreased by increasing G_p in the hard region and increased by increasing G_p in the soft region. Particularly, the transition from hardening to softening behaviours may be controlled by G_p as well as the resonance shift. The associated phase-frequency responses are presented in Fig. 3. As the phase is set to be a nearly to $k\pi$ additive, various configurations with $k\pi$-translation can be presented. For the subharmonic resonances $\omega \approx 2\omega_L$, the numerical results are given by using the following approximation: $\Lambda \approx -F_1/(6\omega_L{}^2)$.

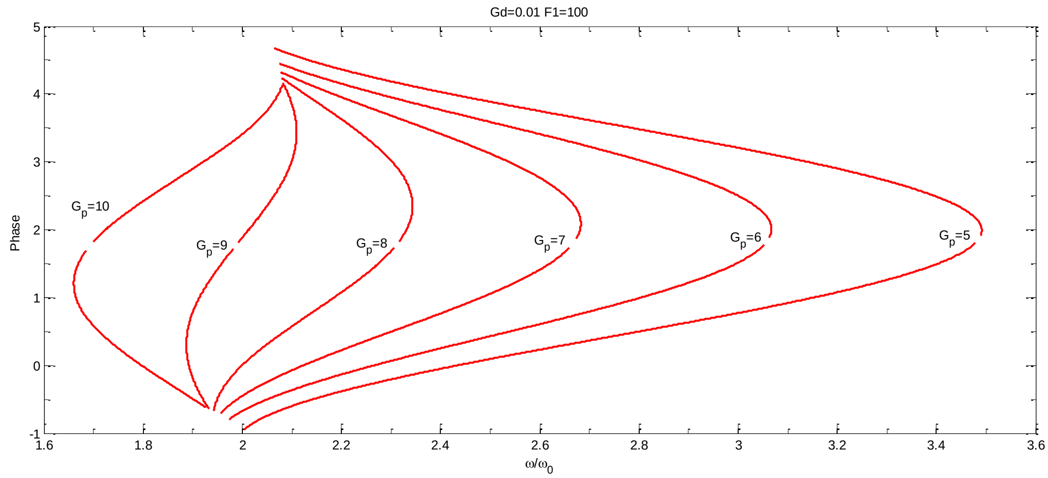

Figure. 5: Subharmonic nonlinear phase-frequency responses $\omega \approx 2\,\omega_L$ for a uniformly excited sandwich S-S beam for various feedback parameters: G_p=5, 6, 7, 8, 9, 10, G_d=0.01 and F_1=100.

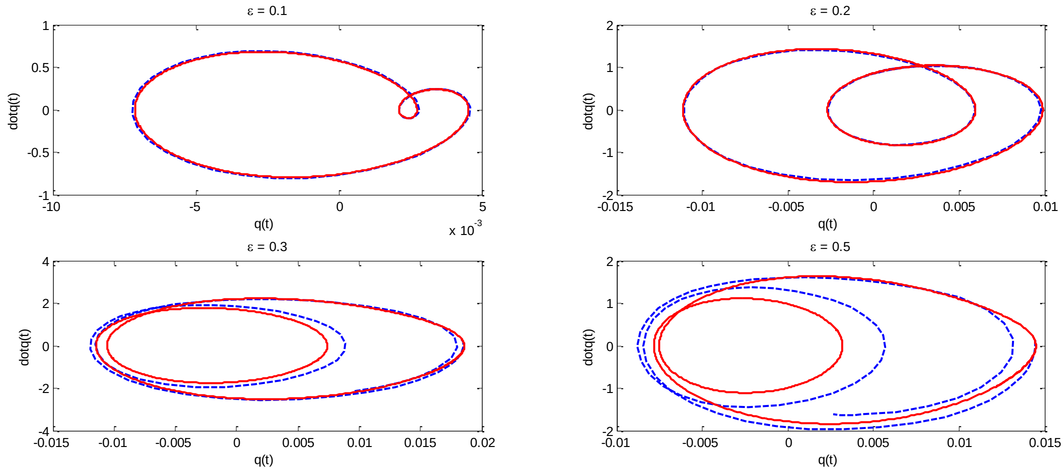

Figure. 6: Phase plane solution for the superharmonic $\omega = 0.51\,\omega_L$ at the small equilibrium solution (a, γ) of the algebraic equations (20, 23) for variant values of ϵ. G_p=7, G_d=0.01, F_1= 100, (____ MSM; ……. RK)

The G_p effects on the frequency-amplitude response are presented in figure 4 and figure 5 presents the corresponding frequency-phase responses for ϵ = 1. Again the transition from hard to soft behaviour and a small translation around $\omega \approx 2\omega_L$ is obtained by varying G_p. For numerical comparisons of the obtained time response of equation (6), the phase-plane computed by the Runge-Kutta algorithm and by equation (24) are presented in Figure.6 for $\omega = 0.51\omega_L$ and various values of ϵ. The amplitude and phase-time dependent $a(t)$ and $\gamma(t)$, corresponding to the equilibrium solution, are given by numerically solving the system (20). The initial conditions for the Runge-Kutta algorithm, $\left(q(0), \dot{q}(0)\right)$, are obtained from equation (24) in which $a(0)$ and $\gamma(0)$ are solution of the algebraic system (21). The initial amplitude and phase $\dot{a}(0)$ and $\dot{\gamma}(0)$ are given by equation (20). This figure chows clearly that the analytical and numerical results are in good agreement for small value of ϵ and particularly at small amplitudes.

4. CONCLUSION

The mathematical modelling of the sub and superharmonic resonances of piezoelectric sandwich beams is done based on a nonlinear partial differential equation, nonlinear potential feedback control and a multiple scales method. Based on the Galerkin's method, a nonlinear differential equation with strong nonlinearities is obtained. For hard excitations, the subharmonic and superharmonic resonances are analysed in the vicinity of $\omega \approx \frac{1}{2}\omega_L$ and $\omega \approx 2\omega_L$. The frequency and phase amplitude dependent relationships as well as the time responses are analytically derived. The feedback parameters effects on the frequency, amplitude and phase of sandwich beams are investigated. The obtained analytical models allow one to easily analyse the suppression, amplification and transition of the considered sub and superharmonic resonances.

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