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Geometrical analysis of local stresses evolution in a two-layered phase composite

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Résumé

Le comportement macroscopique d'un composite stratifié biphasé constitué de deux phases; (élastique et élasto-parfaitement plastique) présente quelques caractéristiques particulières. En plus d'une forte anisotropie, quelques chemins de chargements radiaux correspondent à une réponse linéaire comme pour le cas du modèle de Prager isotrope. Pour étudier ce phénomène, nous présentons ici une analyse basée sur une description géométrique des contraintes locales. Il sera montré que le comportement asymptotique du composite stratifie biphasé soumis à un chargement radial est liée à la stabilité de l'évolution des contraintes locales. Par ailleurs cette analyse permet d'expliciter analytiquement le comportement macroscopique asymptotique du stratifié biphasé.

Abstract

The behaviour of a two phase layered composite consisting in elastic and elastic-perfectly plastic phases exhibits some particular features. Among strong anisotropy, some radial loading paths correspond to linear response as for isotropic Prager model. To study this phenomena, a simple analysis based on a geometrical study of the evolution of internal stresses is presented here. It will be shown that the asymptotic behaviour under radial loading paths for the two-phase layered composite is related to the stability of the evolution of local stresses. Moreover this analysis leads to an explicit formulation of the asymptotic macroscopic behaviour of such composite.

1. INTRODUCTION

Generally speaking kinematics hardening is a phenomenological mechanical model accounting for plastic heterogeneity and the resulting internal stresses (Mughrabi 1988). In its simplest form it leads to the so called Prager model and in the one dimensional case to a linear hardening. From a

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micromechanical point of view this model can be obtained from a homogenization procedure assuming elastic-perfectly plastic phases with uniform plastic strain (Suquet 1997). In most cases this is a very crude approximation resulting for instance from the Voigt's model (Fougères and Sidoroff 1989).
However, for layered structure uniform stress and strain can be obtained as an exact result. Complete closed form homogenization procedures have been performed for different types of constituents. Several authors have been interested into different types of constituents, elastic (Backus 1962), rigid plastic (de Buhan and Salençon 1987), elastic-viscoplastic phase (Li and Weng 1998).
The case of perfectly plastic constituents was studied by (El Omri and al 1991) for two phases and in (El Omri and al, 2000) for any number of elastic plastic phases. In all of these cases, the illustrated responses have shown some important features. Typical result, given in Fig. 2, represents the off axis tensile behavior of a layered composite when only one phase is elastic where the second is elastic perfectly plastic:

a) Very strong anisotropy, with the in axis tensile behavior ($\theta = 0°$ and $\theta = 90°$), close to the Voigt's approximation while the off axis behavior appear much weaker.

b) Significant non-linear hardening (one plastified phase). This non linearity strongly depends on the orientation of the layers (linear behavior for $\theta = 0°$, weakly non linear behavior for $\theta = 90°$ and strongly non linear for intermediate values).

c) It may even happen ($\theta = 45°$, for instance) that the plastic limit occurs with only one plastified phase.

The purpose of this paper is to explain all of these unpredictable features. Attention however will be restricted to only one single plastic phase, the other remains elastic. A detailed analysis based on a geometrical description of the evolution of local stresses is proposed to clarify all of the features of this model under a general radial stress loading-path. By this study it will lead to the explicit asymptotic behavior of such composite.

2. MECHANICAL FRAMEWORK

The investigated composite consists in a periodical layered superposition of two different materials A and B with volume fractions c and (1 – c) respectively. One material (A) is isotropic elastic-perfectly plastic obeying von Mises yielding condition, while the second (B) has the same elastic properties as (A) (elastic homogeneity)

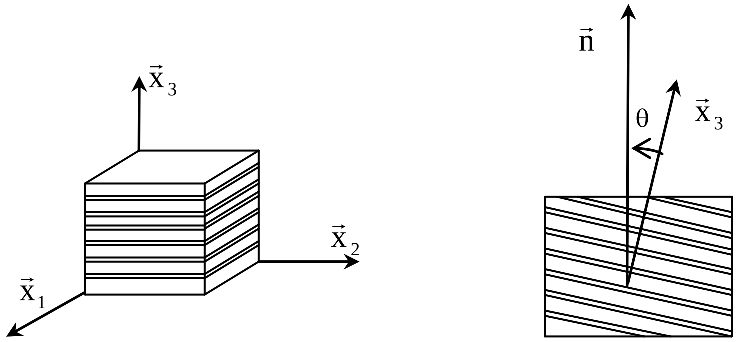

Figure 1 : Layered structure and off axis tensile test direction

The elastic law in both phases is

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon^e_{ij} + \frac{\nu}{1-2\nu}\varepsilon^e_{kk}\delta_{ij}\right), \qquad \varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^P_{ij}. \tag{1}$$

σ, ε, ε^p and ε^e denote respectively stress, total strain, plastic strain and elastic strain. Repeated indices imply sum.

The plastic strain ε^P_{ij} is zero in the elastic phase B while in phase A it is governed by the standard von Mises plastic rule

$$f(\sigma) = \sqrt{s^A_{mn} s^A_{mn}} - \sqrt{\frac{2}{3}}y, \qquad s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{mm}. \tag{2}$$

$$\begin{cases} d\varepsilon^{PA}_{ij} = 0 \quad \text{if} \quad f(\sigma) < 0 \\ d\varepsilon^{PA}_{ij} = \dfrac{d\lambda}{\sqrt{s^A_{mn} s^A_{mn}}} s^A_{mn} \quad \text{if} \quad f(\sigma) = 0, \quad d\lambda \geq 0 \end{cases} \tag{3}$$

Macroscopic behavior of the composite relates macroscopic stress Σ and strain $\in$ given as a mean of their local counterparts:

$$\Sigma = c\sigma^A + (1-c)\sigma^B, \quad \in = c\varepsilon^A + (1-c)\varepsilon^B \tag{4}$$

In the layered case however, they are easily obtained from continuity and equilibrium as:

$$\begin{cases} \sigma^A_{i3} = \sigma^B_{i3} = \Sigma_{i3} & i = 1,2,3. \\ \varepsilon^A_{\alpha\beta} = \varepsilon^B_{\alpha\beta} = \in_{\alpha\beta} & \alpha,\beta = 1,2. \end{cases} \tag{5}$$

As shown in (El Omri and Sidoroff 1991), the localization rule for σ^A_{ij} and the plastic multiplicator $d\lambda$ are obtained from the consistency condition

$$\begin{cases} \sigma_{11}^{A}=\Sigma_{11}-\dfrac{(1-c)E}{(1-\nu^2)}(\varepsilon_{11}^{PA}+\nu\varepsilon_{22}^{PA}) \quad \sigma_{22}^{A}=\Sigma_{22}-\dfrac{(1-c)E}{(1-\nu^2)}(\nu\varepsilon_{11}^{PA}+\varepsilon_{22}^{PA}) \\ \sigma_{12}^{A}=\Sigma_{12}-\dfrac{(1-c)E}{(1+\nu)}\varepsilon_{12}^{PA} \end{cases} \tag{6}$$

$$\frac{d\lambda}{\sqrt{s_{mn}^{A}s_{mn}^{A}}}=\frac{(1-\nu^2)}{(1-c)E}\frac{s_{mn}^{A}\,d\Sigma_{mn}}{s_{11}^{A\,2}+s_{22}^{A\,2}+2\,\nu s_{11}^{A}\,s_{22}^{A}+2(1-\nu)s_{12}^{A\,2}} \tag{7}$$

Finally combination of all these equations results in

$$d\in_{ij}=\frac{(1+\nu)}{E}d\Sigma_{ij}-\frac{\nu}{E}\delta_{ij}d\Sigma_{kk}+\frac{c(1-\nu^2)}{(1-c)E}\frac{s_{mn}^{A}\,d\Sigma_{mn}}{s_{11}^{A\,2}+s_{22}^{A\,2}+s_{11}^{A}\,s_{22}^{A}+2(1-\nu)s_{12}^{A\,2}}s_{ij}^{A} \tag{8}$$

These differential equations describe the elastic plastic behavior of the investigated material under prescribed macroscopic stress loading path.

In the case of the off axis tensile test (fig. 1), saying a simple tensile test in some direction $\vec{n}$, The stress tensor components are given by

$$\Sigma_{ij}=\Sigma\, n_i n_j$$

Integration of this equation can be performed numerically (El Omri and all, 2000). The longitudinal stress versus the longitudinal strain is represented in Fig. 2 for the material constants (ν=0.35, y=20.7 MPa, c = 0.9)

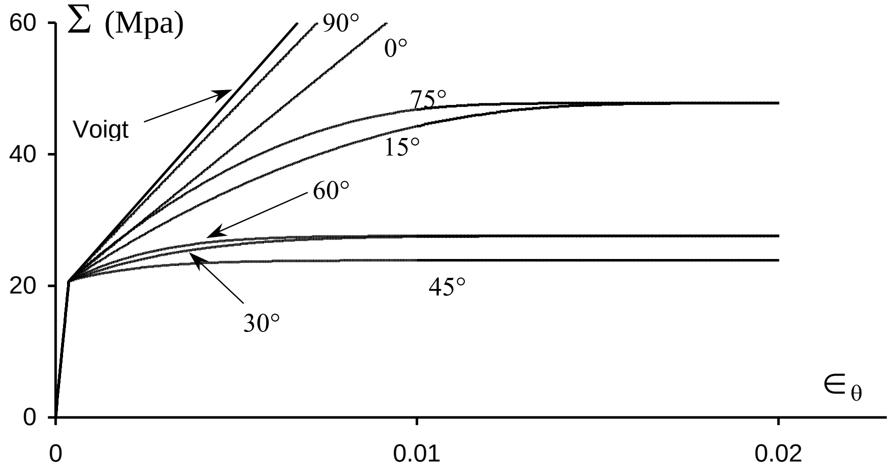

Figure 3 : Off axis tensile test behavior for different values of θ

3. GEOMETRICAL REPRESENTATION

In the following, the analysis is limited to the case where the composite is submitted to a radial stress loading path

$$\Sigma = \Sigma\ a \tag{9}$$

where the tensor a defines a radial stress path.

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

The study presented here is based on a geometrical description of the local deviatoric stress evolution. For this purpose, any deviatoric tensor **t** can be decomposed in an appropriate basis, by using the following setting (El Omri Sidoroff.2000.)

$$\begin{array}{lll} t_I = \dfrac{(t_{11} + t_{22})}{\sqrt{2}} & t_{II} = \dfrac{(t_{11} - t_{22})}{\sqrt{2}} & t_{III} = \sqrt{2}\, t_{12} \\ t_{IV} = \sqrt{2}\, t_{13} & t_V = \sqrt{2}\, t_{23} & t_{VI} = -\sqrt{2}\, t_I^D \end{array} \tag{10}$$

$$t_I^D = \frac{(2t_{33} - t_{11} - t_{22})}{3\sqrt{2}} \qquad t_K^D = t_K \quad K = II..VI$$

By taking into account the localisation condition (5), von Mises yield criterion related to elastic plastic phase reads :

$$3\, \sigma_I^{D^2} + \sigma_{II}^{\ 2} + \sigma_{III}^{\ 2} + \Sigma_{IV}^{\ 2} + \Sigma_V^{\ 2} = \frac{2}{3} y^2 \tag{11}$$

While the evolution of the local deviatoric stress is obtained using (3) and (7)

$$d\sigma_I^D = d\Sigma_I^D - \frac{3\sigma_I^D d\Sigma_I^D + \sigma_{II} d\Sigma_{II} + \sigma_{III} d\Sigma_{III} + \sigma_{IV} d\Sigma_{IV} + \sigma_V d\Sigma_V}{(1+\nu){\sigma_I^D}^2 + (1-\nu){\sigma_{II}}^2 + (1-\nu){\sigma_{III}}^2} \frac{1}{3}(1-\nu)\sigma_I^D$$

$$d\sigma_{II} = d\Sigma_{II} - \frac{3\sigma_I^D d\Sigma_I^D + \sigma_{II} d\Sigma_{II} + \sigma_{III} d\Sigma_{III} + \sigma_{IV} d\Sigma_{IV} + \sigma_V d\Sigma_V}{(1+\nu){\sigma_I^D}^2 + (1-\nu){\sigma_{II}}^2 + (1-\nu){\sigma_{III}}^2} (1+\nu)\sigma_{II} \tag{12}$$

$$d\sigma_{III} = d\Sigma_{III} - \frac{3\sigma_I^D d\Sigma_I^D + \sigma_{II} d\Sigma_{II} + \sigma_{III} d\Sigma_{III} + \sigma_{IV} d\Sigma_{IV} + \sigma_V d\Sigma_V}{(1+\nu){\sigma_I^D}^2 + (1-\nu){\sigma_{II}}^2 + (1-\nu){\sigma_{III}}^2} (1+\nu)\sigma_{III}$$

3.1 The triaxial case

To study the stability of the last equations, let us first consider the case where **a** is diagonal

$$a = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \tag{13}$$

The von Mises yield condition in the plastified case follows from eqn (11) as

$$3{\sigma_I^D}^2 + {\sigma_{II}}^2 = \frac{2}{3}y^2 \tag{14}$$

This suggests a representation of the local stress state by a point P in the plane (X_1, X_2)

$$X_1 = \sqrt{3}\sigma_I^D, \quad X_2 = \sigma_{II} \tag{15}$$

so that the plastic yield surface is the circle C of radius $\sqrt{\frac{2}{3}}y$ and the stress state in plastic range is entirely described by the polar angle α (Fig. 3)

$$X_1 = \sqrt{\frac{2}{3}y^2}\cos\alpha, \quad X_2 = \sqrt{\frac{2}{3}y^2}\sin\alpha \tag{16}$$

The evolution of P will be therefore obtained from the evolution of α. Substituting eqns (13), (15), (16) in (12), it follows :

$$\frac{d\alpha}{d\Sigma}=\frac{\sqrt{3}\left(\frac{(1+\nu)^2}{3}a_{II}{}^2+9(1-\nu)^2 a_I^{D\;2}\right)\sin(\alpha-\alpha_0)}{(1+\nu)\cos^2\alpha+3(1-\nu)\sin^2\alpha} \tag{17}$$

where α_0 is given by

$$\begin{cases} \cos\alpha_0 = 3(1-\nu)a_I^D \dfrac{1}{\left(9(1-\nu)^2 a_I^{D^2}+1/3(1+\nu)^2 a_{II}{}^2\right)} \\ \sin\alpha_0 = \dfrac{(1+\nu)}{\sqrt{3}}a_{II}\dfrac{1}{\left(9(1-\nu)^2 a_I^{D^2}+1/3(1+\nu)^2 a_{II}{}^2\right)} \end{cases}$$

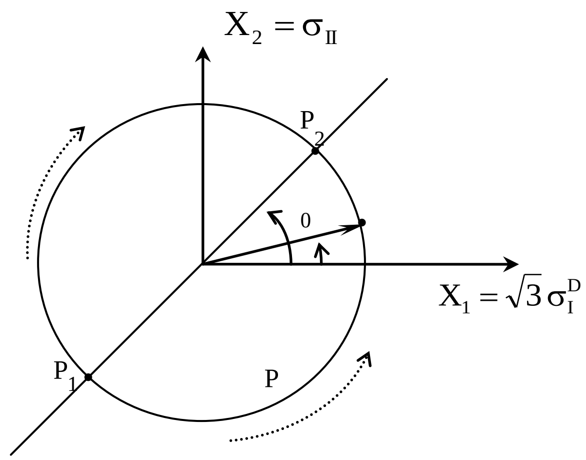

Figure 3 : Local stress state representative point in the triaxial case

This implies that for any given initial value of α different from α_0, the point P tends asymptotically to P_2 while P_1 is clearly instable.
By using (15) and (16), the equation (17) gives the asymptotic values σ_I^D and σ_{II}

$$\sigma_I^D=\frac{\sqrt{3}(1-\nu)a_I^D}{\sqrt{\frac{(1+\nu)^2}{3}a_{II}{}^2+9(1-\nu)^2 a_I^{D\;2}}} \qquad \sigma_{II}=\frac{\frac{1}{\sqrt{3}}(1+\nu)a_{II}}{\sqrt{\frac{(1+\nu)^2}{3}a_{II}{}^2+9(1-\nu)^2 a_I^{D\;2}}} \tag{18}$$

The homogenized response corresponding to this stationary state is obtained by substituting eqn (18) in eqn (8),

$$\begin{bmatrix} d\in_I \\ d\in_{II} \\ d\in_{VI} \end{bmatrix} = \frac{d\Sigma}{E}\left(\begin{bmatrix} 1-\nu & 0 & -\sqrt{2}\nu \\ 0 & 1+\nu & 0 \\ -\sqrt{2}\nu & 0 & 1+\nu \end{bmatrix} + \frac{c(1-\nu)}{1-c}\begin{bmatrix} 1 & 0 & -\sqrt{2} \\ 0 & \frac{1+\nu}{1-\nu} & 0 \\ -\sqrt{2} & 0 & -2 \end{bmatrix}\right)\begin{bmatrix} a_I \\ a_{II} \\ a_{VI} \end{bmatrix} \tag{19}$$

implying a linear response, i.e., an asymptotic linear hardening.
The stress evolution during the hardening therefore is described by the evolution of point P from its initial position at the onset of plasticity to its asymptotic stationary state P_2 in the considered triaxial loading path, and due to elastic homogeneity the stress state is homogeneous in initial elastic domain ($\sigma_{ij} = \Sigma_{ij}$). Plasticity will then be initiated when the von Mises yield condition is satisfied

$$tg\alpha = \sqrt{3}\,\frac{a_I^D}{a_{II}} \tag{20}$$

Comparison of (20) with (17) shows that there will be an evolution of P, and therefore a non linear hardening unless one of the three following conditions are satisfied

$$\nu = \frac{1}{2} \text{ or } a_I^D = 0 \text{ or } a_{II} = 0 \tag{21}$$

In which case P coincides with P2 , resulting in the linear hardening rule (19).

3.2 The general loading path case

Let us now consider the general case of loading direction. Through proper choice of the ($\vec{x}_1, \vec{x}_2$) axis in the layer plane, **a** can be chosen, without restriction, as

$$\Sigma = \Sigma\begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{22}$$

The component a_{12} is taken as zero due to transverse isotropy of the material. The von Mises condition (11) then reads :

$$3{\sigma_I^D}^2 + \sigma_{II}^2 + \Sigma^2 ({a_{IV}}^2 + {a_V}^2) = \frac{2}{3}y^2 \tag{23}$$

This equation corresponds to a sphere of radius $\sqrt{\frac{2}{3}}y$ in the (X_1, X_2, X_3) space

$$X_1 = \sqrt{3}\sigma_I^D,\ \ X_2 = \sigma_{II},\ \ X_3 = \Sigma\sqrt{{a_{IV}}^2 + {a_V}^2}. \tag{24}$$

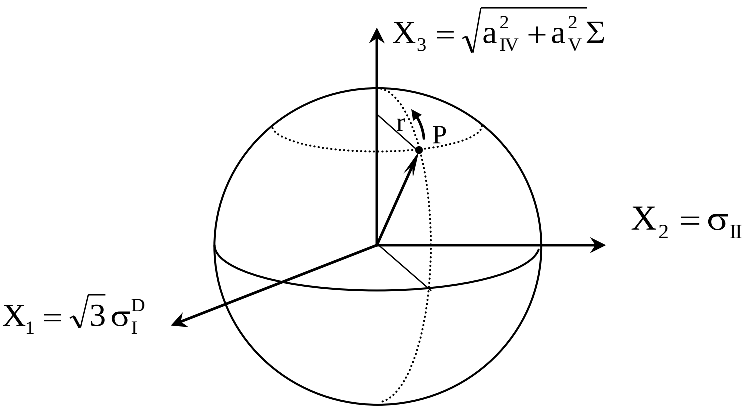

Figure 4 : Local stress state representative point in the general case

For a fixed value of Σ therefore, X_3 is fixed so that the representative point (Fig. 4) belongs in a horizontal plane to a circle of radius r

$$r = \sqrt{\frac{2}{3}\,y^2 - (a_{IV}{}^2 + a_V{}^2)\,\Sigma^2} \tag{25}$$

which depends on Σ and decreases with increasing Σ. This radius vanishes for

$$\Sigma^2 = \frac{1}{(a_{IV}{}^2 + a_V{}^2)}\,\frac{2}{3}\,y^2 \tag{26}$$

so that the representative point tends to the pole of the sphere so that σ_I^D and σ_{II} also vanish. This corresponds to the plastic yield limit.

4. OFF AXIS TENSILE BEHAVIOR

Let us now come back to the off axis tensile behavior observed in section II and Fig. 2. The loading direction **a** in this case is

$$a = \begin{bmatrix} \sin^2\theta & 0 & \cos\theta\sin\theta \\ 0 & 0 & 0 \\ \cos\theta\sin\theta & 0 & \cos^2\theta \end{bmatrix} \tag{27}$$

so that in the basis (10)

$$a_I = a_{II} = \frac{\sin^2\theta}{\sqrt{2}}, \quad a_{IV} = \sqrt{2}\sin\theta\cos\theta, \quad a_{VI} = \cos^2\theta,$$
$$a_I^D = \frac{1}{3}\left(\frac{\cos^2\theta}{\sqrt{2}} - \sqrt{2}\sin^2\theta\right). \tag{28}$$

Similarly the longitudinal strain $\in_\theta$ is defined by

$$d\in_\theta = d\in_{ij} n_i n_j = \sin^2\theta\, d\in_{11} + \cos^2\theta\, d\in_{33} + \sin 2\theta\, d\in_{13}$$
$$d\in_\theta = \frac{1}{\sqrt{2}}\sin^2\theta\,(d\in_I + d\in_{II})\ \cos^2\theta\, d\in_{VI} + \frac{1}{\sqrt{2}}\sin 2\theta\, d\in_{IV} \tag{29}$$

The triaxial case is obtained only for the in axis tensile test (θ= 0°or 90°). For θ= 0°, i.e., for a specimen normal to the layers, a_{II} vanishes so that as discussed above a linear hardening is obtained with ($\alpha=\alpha_0=0$). The linear macroscopic response being

$$\frac{d\Sigma}{d\in} = \beta E \qquad \beta = \frac{(1-c)}{1+c(1-2\nu)} \tag{30}$$

For θ = 90°, the behavior still tends to the linear hardening given by eqn (19) for large $\in$,

$$\frac{d\Sigma}{d\in} = \beta^\infty E \qquad \beta^\infty = 1-c \tag{31}$$

but with non linear hardening starting from an initial slope after yield

$$\left.\frac{d\Sigma}{d\in}\right|_{\varepsilon^P=0} = \beta^0 E \qquad \beta^0 = \frac{(1-c)(5-4\nu)}{(5-4\nu)-c(1-4\nu+4\nu^2)} \tag{32}$$

All this values may be compared to the Voigt's approximation that corresponds to a linear hardening

$$\frac{d\Sigma}{d\in} = \beta^V E \qquad \beta^V = \frac{3(1-c)}{3-c(1-2\nu)} \tag{33}$$

For instance in the numerical example which corresponds to Fig. 2

(ν=0.35, y=20.7 MPa, c = 0.9)

$$\beta^V = 0.10989 \quad \beta = 0.78740 \quad \beta^\infty = 0.1 \quad \beta^0 = 0.102302 \tag{34}$$

The general case investigated in section VI corresponds to off axis tensile test (0°<θ< 90°) with non vanishing a_{IV}. Accordingly a plastic limit stress is obtained from (26)

$$\Sigma = \frac{1}{\sqrt{3}} \frac{y}{\sin\theta\cos\theta}, \tag{35}$$

which is minimum for θ = 45° and which assumes equal values for θ= 0° and for θ= 90°. This explains what is observed in Fig. 2. The initial slope after yield of the corresponding stress strain curve

$$\frac{d\Sigma}{d \in} = \beta_0(\theta)\, E \tag{36}$$

is easily computed and the corresponding curve is represented in Fig. 5.

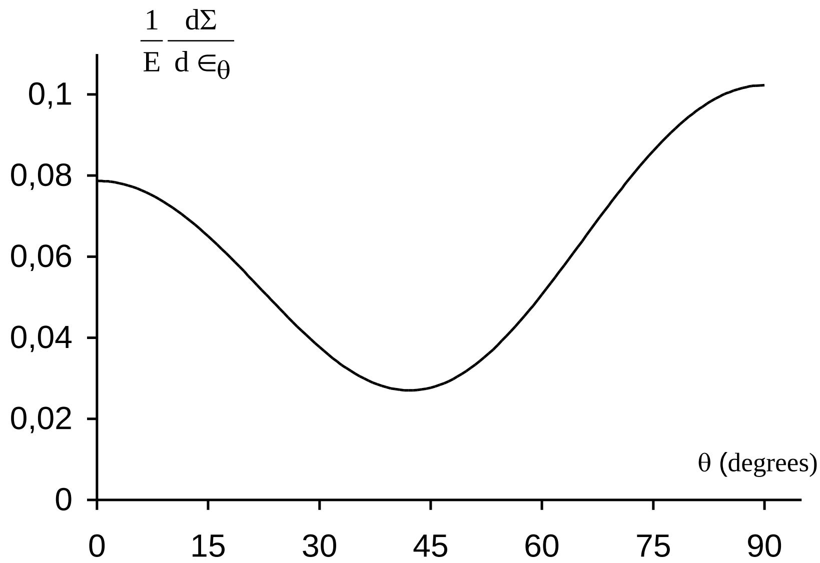

Figure 5 : Initial slope after yield of stress strain curve in off axis tensile test.

5. CONCLUSION

The geometrical analysis presented here leads to an analytic and quantitative description of the behaviour of a two-phase layered structure. The asymptotic behaviour of such composite is obtained explicitly. It correspond for every case to a stable state of the evolution of the local stresses. All of the paths corresponding to linear responses (linear hardening) are identified precisely.
Although it allows an in depth understanding of the illustrated responses (Fig. 2), this analysis is limited to two isotropic constituents and to radial stress path tests. Its generalization to other two phase elastic plastic composite remains difficult. An analysis based upon thermodynamic formulation of the model associated to the considered heterogeneous material is under works and seems more suitable for generalisation.

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