

## **Anisotropic Prager's model resulting from micro-heterogeneity: A case of two - phase fibrous composite**

A. El Omri

Laboratoire de Mécanique des Milieux Hétérogènes  
Faculté des Sciences et Techniques, BP 416, Tanger , Maroc

F. Sidoroff

L.T.D.S, Ecole Centrale de Lyon. UMR C 5513. BP 163, 69131  
Ecully, France.

A. Hihi

Laboratoire de Mécanique des Matériaux,  
Faculté des Sciences, BP 1014 Agdal, Rabat , Maroc

### **Résumé**

Le but de ce travail est d'établir les bases d'une formulation explicite des modèles anisotropes généralisés correspondant à des matériaux composites à structures géométriques simples comme les composites stratifiés ou fibreux. D'abord le comportement macroscopique sera obtenu par une méthode d'homogénéisation décomposée. Ensuite, l'identification explicite des modèles standards généralisés sera donnée par une démarche thermodynamique. On s'intéressera tout particulièrement aux tenseurs constitutifs relatifs au comportement global d'un composite fibreux.

### **Abstract**

The purpose of the present work is to lay the basis for the explicit formulation of generalized anisotropic models corresponding to some composite materials with simple geometrical structures as layered and fibrous structure. At first the macroscopic behavior will be obtained using a decomposed homogenization procedure. Then, explicit elastic plastic anisotropic models of the investigated structures will be given and identified within a thermodynamic framework. Attention will be focused on special fibrous composites for which structural tensors defining the overall behavior are given explicitly.

### **1. INTRODUCTION**

Most aspects of the elastic-plastic behavior of metals under cyclic loading result from the various levels of microheterogeneity in the material. Elastic-plastic homogenization therefore appears as an appropriate tool for the construction and identification of cyclic plasticity model. The concept of kinematic hardening for instance clearly finds its justification here. Similarly the Masing model, which can be interpreted as resulting from the Voigt's homogenization [1], reflects the essential features of the hysteresis observed in cyclic loading.

From a qualitative point of view, the homogenized elastic-plastic behavior of a multiphase material consisting in a mixture of different elastic or perfectly plastic materials is well understood. Under increasing proportional loading, for instance, an elastic response is first observed until the first local

plastification occurs. This is followed by a macroscopic hardening resulting from the extension of the plastic zone, eventually ending up with a global yielding for complete plastification. From a quantitative point of view, however very little is known. The Voigt's approximation mentioned earlier cannot be considered as satisfactory. In particular it ignores the anisotropy resulting from the geometrical structure of the heterogeneity. Numerical simulations of course, can be performed but they must be repeated for each loading path and if they may give some physical insight, they will not be of much help for constructing explicit homogenized constitutive equation.

From a theoretical point of view, much work has been devoted to the general aspects of elastic plastic (or viscoplastic) homogenization as well as to the analysis of some special cases. The essential features of the resulting model are now reasonably established: generalized standard structure [2,3,4].

The purpose of the present work is to lay the basis for the explicit formulation of such models for some periodic cases corresponding to the simplest geometrical and material structures.

## 2. GENERAL FRAMEWORK

### 2.1. Local and global behavior

A heterogeneous elastic perfectly plastic or viscoplastic material is concerned within the framework of generalized standard model. The local constitutive equation at each point of the heterogeneous medium can then be written from the elastic energy  $w(\varepsilon^e)$  and dissipation potential  $\Omega(\sigma)$ :

$$\begin{aligned} w(\varepsilon^e) &= \frac{1}{2} \varepsilon^e : \mathbf{a} \varepsilon^e, \quad \varepsilon = \varepsilon^e + \varepsilon^p \\ \phi &= \sigma : d\varepsilon^p \geq 0 \\ \sigma &= \frac{\partial w(\varepsilon^e)}{\partial \varepsilon^e} = \mathbf{a} \varepsilon^e, \quad d\varepsilon^p \in \partial\Omega(\sigma) \end{aligned} \quad (1)$$

where  $\mathbf{a}$  denotes the rigidity of the medium. Stress tensor, total strain, elastic and plastic strain are denoted respectively by  $\sigma, \varepsilon, \varepsilon^e$  and  $\varepsilon$ . While  $\partial\Omega(\sigma)$  denotes the subdifferential of the convex dissipation potential  $\Omega(\sigma)$  and reduces to the usual derivative  $\frac{\partial\Omega(\sigma)}{\partial\sigma}$  in the viscoplastic case and to the usual associated flow rule in the rate independent plastic case

$$d\varepsilon^p = \begin{cases} 0 & \text{if } f(\sigma) \leq 0 \\ d\lambda \frac{\partial f(\sigma)}{\partial \sigma} & \text{if } f(\sigma) = 0 \text{ (} d\sigma \geq 0 \text{)} \end{cases} \quad (2)$$

with yield function  $f(\sigma)$  giving  $\Omega(\sigma)$  as the indicator function of the plastic convex defined by  $f(\sigma) \leq 0$ . For definiteness we shall also assume that the dissipation potential  $\Omega(\sigma)$  or yield function  $f$  are expressed in terms of a Hill type equivalent stress  $\bar{\sigma}$

$$f(\sigma) = \bar{\sigma} - y \quad \text{or} \quad \Omega = \Omega(\bar{\sigma}) \quad \text{with} \quad \bar{\sigma} = \sqrt{\sigma \mathbf{h} \sigma}$$

where  $\mathbf{h}$  is a non-dimensional fourth order tensor. The plastic or viscoplastic flow rule then reduces to

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} = \frac{d\lambda}{\bar{\sigma}} \mathbf{h} \boldsymbol{\sigma}, \quad d\lambda = \sqrt{d\boldsymbol{\varepsilon}^p \mathbf{h} d\boldsymbol{\varepsilon}^p} \geq 0 \quad \text{in the plastic case}$$

or to

$$d\boldsymbol{\varepsilon}^p = \frac{\partial \Omega(\bar{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \quad \text{in the viscoplastic case.}$$

From now on we shall focus on the rate independent plastic case, but the presented analysis readily extends to the viscoplastic case. The local behavior is then entirely defined from the local elastic stiffness  $\mathbf{a}(\mathbf{x})$ , plastic yield shape tensor  $\mathbf{h}(\mathbf{x})$  and yield stress  $y(\mathbf{x})$ . The homogenization problem consists in deriving from this local knowledge, the global constitutive equation relating the macroscopic strain and stress:

$$\boldsymbol{\Sigma} = \langle \boldsymbol{\sigma}(\mathbf{x}) \rangle \quad \boldsymbol{\varepsilon} = \langle \boldsymbol{\varepsilon}(\mathbf{x}) \rangle \quad (3)$$

where  $\langle \varphi(\mathbf{x}) \rangle$  represents an appropriate average of a local quantity  $\varphi(\mathbf{x})$ .

## 2.2 Layered and decomposed homogenization

Formulated in that way, the homogenization problem cannot be solved exactly in the general case. We shall restrict ourselves to a special case in which this can be achieved exactly and explicitly: the "decomposed homogenization" which will be introduced now.

The basic example is the layered homogenization that occurs when the heterogeneity superposition of homogeneous layers perpendicular to  $\bar{x}_3$  direction (Figure 1). It is then easily shown that the localization conditions are

$$\begin{aligned} \sigma_{i3} &= \Sigma_{i3}, \quad i=1,2,3. \\ \varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta}, \quad \alpha,\beta=1,2. \end{aligned} \quad (4)$$

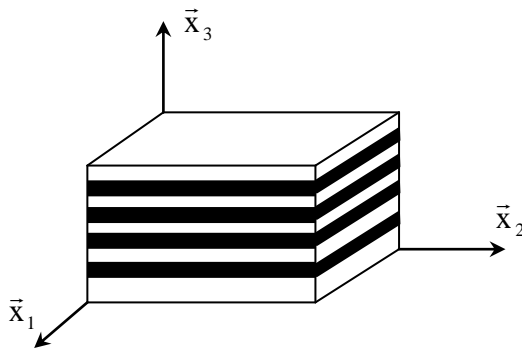


Figure 1: Layered Structure

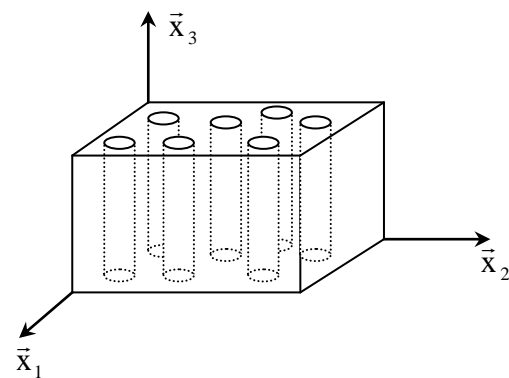


Figure 2: Fibrous structure

This allows a complete analytical solution and the resulting elastic-plastic behavior has been thoroughly analyzed and described in [5] for two phase composite. For many phase this has been studied in [6]. The case of two viscoplastic phase has also been analyzed in [7].

A complete constitutive model yet remains to be formulated and this was the starting point of the work presented here. More generally the decomposed homogenization is defined in the following way:

i.) There exists a decomposition of the vector space  $\mathfrak{T}$  of all symmetric tensors into two orthogonal subspaces

$$\mathfrak{T} = \mathfrak{T}_L \oplus \mathfrak{T}_Q \quad \mathbf{t} = \mathbf{t}_L + \mathbf{t}_Q \quad (5)$$

ii.) The localization conditions can be written as:

$$\boldsymbol{\varepsilon}_Q = \boldsymbol{\varepsilon}_Q \quad \boldsymbol{\sigma}_L = \boldsymbol{\Sigma}_L \quad (6)$$

This is obviously the case for layered homogenization where (4) can be written by taking:

$$\mathbf{t}_L = \begin{pmatrix} 0 & 0 & t_{13} \\ 0 & 0 & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix} \quad \mathbf{t}_Q = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7)$$

Strictly speaking this seems to be the only exact case, but this can also be used as a reasonable approximation: for instance (7) can also be used as an approximation in the case of flat grains or particles. Another important application is the case of a fibrous structure in the  $\bar{x}_3$  direction for which an approximate homogenization [8,9] is often obtained from

$$\boldsymbol{\varepsilon}_Q = \boldsymbol{\varepsilon}_Q \quad \boldsymbol{\sigma}_L = \boldsymbol{\Sigma}_L \quad \mathbf{t}_L = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & 0 \end{pmatrix} \quad \mathbf{t}_Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_{33} \end{pmatrix} \quad (8)$$

It should also be noted that the Voigt ( $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$ ) and Reuss ( $\boldsymbol{\sigma} = \boldsymbol{\Sigma}$ ) approximations can also be considered as limiting cases.

### 2.3. Decomposed elastic homogenization

As presented in [4] the basic idea for decomposed homogenization is to transform the constitutive equation in a hybrid form relating  $(\boldsymbol{\sigma}_Q, \boldsymbol{\varepsilon}_L)$  to  $(\boldsymbol{\varepsilon}_Q, \boldsymbol{\sigma}_L)$  so that according to (3), homogenization reduces to averaging.

It also follows from (1) and (2) that, if the material is constituted from the mixture of a finite number of homogeneous constituents (multiphase materials) the stress and the strain are homogeneous in each phase  $\alpha$ , so that (4) and (6) results in:

$$\begin{aligned} \boldsymbol{\sigma}_L = \boldsymbol{\Sigma}_L \quad \boldsymbol{\Sigma}_Q = \langle \boldsymbol{\sigma}_Q \rangle &= \sum_{\alpha} \mathbf{c}_{\alpha} \boldsymbol{\sigma}_Q^{\alpha}, \\ \boldsymbol{\sigma}_Q = \boldsymbol{\varepsilon}_Q \quad \boldsymbol{\varepsilon}_L = \langle \boldsymbol{\varepsilon}_L \rangle &= \sum_{\alpha} \mathbf{c}_{\alpha} \boldsymbol{\varepsilon}_L^{\alpha} \end{aligned} \quad (9)$$

with  $\mathbf{c}_{\alpha}$  being the volume fraction of the constituent  $\alpha$ . This will be assumed in the following, although the extension to a continuous heterogeneity would be straightforward. As a first example and

for further reference we shall consider the case of elastic homogenization. The hybrid local constitutive equation obtained from partial inversion of the usual elastic form is

$$\begin{bmatrix} \sigma_Q \\ \varepsilon_L \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{QQ} & \mathbf{k}_{QL} \\ \mathbf{k}_{LQ} & \mathbf{k}_{LL} \end{bmatrix} \begin{bmatrix} \varepsilon_Q \\ \Sigma_L \end{bmatrix} \quad (10)$$

$$\mathbf{k}_{QQ} = \mathbf{a}_{QQ} - \mathbf{a}_{QL} \mathbf{a}_{LL}^{-1} \mathbf{a}_{LQ}, \quad \mathbf{k}_{LL} = \mathbf{a}_{LL}^{-1}, \quad \mathbf{k}_{QL} = \mathbf{a}_{QL} \mathbf{a}_{LL}^{-1} = -\mathbf{a}_{LQ}^T.$$

By averaging (10), the homogenized elastic law is obtained as:

$$\begin{bmatrix} \Sigma_Q \\ \varepsilon_L \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{QQ} & \mathbf{K}_{QL} \\ \mathbf{K}_{LQ} & \mathbf{K}_{LL} \end{bmatrix} \begin{bmatrix} \varepsilon_Q \\ \Sigma_L \end{bmatrix} \quad \mathbf{K}_{QQ} = \langle \mathbf{k}_{QQ} \rangle \text{ etc...} \quad (11)$$

which partially inverted back results in a generalized mixture law which can be explicitly written [4]:

$$\Sigma = \mathbf{A} \varepsilon \quad (12)$$

with

$$\mathbf{A}_{QQ} = \mathbf{K}_{QQ} - \mathbf{K}_{QL} \mathbf{K}_{LL}^{-1} \mathbf{K}_{LQ}; \quad \mathbf{A}_{LL} = \mathbf{K}_{LL}^{-1}; \quad \mathbf{A}_{QL} = \mathbf{K}_{QL} \mathbf{K}_{LL}^{-1}$$

Similarly the local stress is given, by combining (10) and (11), as

$$\sigma = \mathbf{C} \Sigma; \quad \mathbf{C} = \begin{bmatrix} \mathbf{k}_{QQ} \mathbf{K}_{QQ}^{-1} & \mathbf{k}_{QL} - \mathbf{k}_{QQ} \mathbf{K}_{QQ}^{-1} \mathbf{K}_{QL} \\ \mathbf{0} & \mathbf{I}_{LL} \end{bmatrix} \quad (13)$$

## 2.4 Decomposed elastic plastic homogenization

The local elastic law is written in hybrid form by replacing  $\varepsilon$  in (10) by  $(\varepsilon - \varepsilon^p)$ . This law can then be expressed in the following form:

$$\begin{bmatrix} \sigma_Q \\ \varepsilon_L \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{QQ} & \mathbf{k}_{LQ} \\ \mathbf{k}_{LQ} & \mathbf{k}_{LL} \end{bmatrix} \begin{bmatrix} \varepsilon_Q \\ \Sigma_L \end{bmatrix} + \begin{bmatrix} -\mathbf{k}_{QQ} \varepsilon_Q^p \\ \varepsilon_L^p - \mathbf{k}_{LQ} \varepsilon_Q^p \end{bmatrix} \quad (14)$$

When averaged, this gives

$$\begin{bmatrix} \Sigma_Q \\ \varepsilon_L - \varepsilon_L^p \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{QQ} & \mathbf{K}_{QL} \\ \mathbf{K}_{LQ} & \mathbf{K}_{LL} \end{bmatrix} \begin{bmatrix} \varepsilon_Q - \varepsilon_Q^p \\ \Sigma_L \end{bmatrix} \quad (15)$$

with

$$\begin{aligned}\epsilon_Q^p &= \langle \mathbf{k}_{QQ} \mathbf{K}_{QQ}^{-1} \epsilon_Q^p \rangle \\ \epsilon_L^p &= \langle \epsilon_L^p - (\mathbf{k}_{LQ} - \mathbf{k}_{LQ} \mathbf{K}_{QQ}^{-1} \mathbf{K}_{QQ}) \epsilon_Q^p \rangle\end{aligned}$$

which reduces to the averaged plastic strain  $\langle \epsilon^p \rangle$  in case of homogeneous elasticity. It should also be noted that using the stress localization tensor (13) for the elastic-plastic phase, this relation could be written in the more usual form:

$$\epsilon^p = \langle \mathbf{C}^T \epsilon^p \rangle \quad (16)$$

Stress localization requires the computation of the local stress  $\sigma$  in the plastic phase. Using (14) and (15) one has:

$$\sigma = \mathbf{C}\Sigma - \mathbf{M}i^p, \quad \mathbf{M} = \begin{bmatrix} \mathbf{k}_{QQ} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (17)$$

where  $i^p$  denotes the plastic incompatibility tensor

$$i^p = \epsilon^p - \epsilon^p$$

To have the plastic evolution laws decomposed in  $\mathfrak{S}_L \oplus \mathfrak{S}_Q^\perp$  one must formulate the plastic multiplier  $d\lambda$  that is obtained by consistency condition ( $df(\sigma) = 0$ ):

$$\frac{\partial f(\sigma_Q, \Sigma_L)}{\partial \sigma_Q} d\sigma_Q + \frac{\partial f(\sigma_Q, \Sigma_L)}{\partial \Sigma_L} d\Sigma_L = 0 \quad (18)$$

By using (14) and the constitutive equation (2) one has:

$$d\sigma_Q = \mathbf{k}_{QQ} d\epsilon_Q + \mathbf{k}_{QL} d\Sigma_L - d\lambda \mathbf{k}_{QQ} \frac{\partial f(\sigma_Q, \Sigma_L)}{\partial \sigma_Q} \quad (19)$$

Combining these two equations one has explicitly the plastic multiplier

$$d\lambda = \frac{\frac{\partial f}{\partial \sigma_Q} \mathbf{k}_{QQ} d\epsilon_Q + \left( \frac{\partial f}{\partial \sigma_Q} \mathbf{k}_{QL} + \frac{\partial f}{\partial \Sigma_L} \right) d\Sigma_L}{\frac{\partial f}{\partial \sigma_Q} \mathbf{k}_{QQ} \frac{\partial f}{\partial \sigma_Q}} \quad (20)$$

leading so to the incremental macroscopic behavior of such material. The viscoplastic case is treated similarly.

Hence the behavior of such a material is given in a hybrid formulation leading to an expression of any local law independently from the others. The homogenized incremental behavior, written in a hybrid form, can then be obtained only by applying the mean operator. However if this formulation deals with decomposed elastic plastic homogenization it remains complicate and doesn't give other information nor on the macroscopic anisotropy of the investigate heterogeneous material, neither on the nature of the kinematic hardening expected in these cases.

### 3. ANISOTROPIC PRAGER'S MODEL

#### 3.1 Standard generalized formulation

The first material structure investigated will consist in a mixture of one single elastic-plastic phase with any number of elastic phases. This introduces one single internal variable so that a macroscopic Prager's model can be expected to result from the homogenization process. In the general anisotropic case, this model can be formulated from an elastic energy

$$w(\epsilon^e) = \frac{1}{2} \epsilon^e \mathbf{A} \epsilon^e + \frac{1}{2} \epsilon^p \mathbf{B} \epsilon^p \quad (21)$$

$$\Sigma = \mathbf{A} \epsilon^e \quad \phi = (\Sigma - \mathbf{B} \epsilon^p) \cdot d \epsilon^p = Q \cdot d \epsilon^p \geq 0$$

and from a yield function

$$F(\bar{Q}) = \bar{Q} - y \quad \bar{Q} = \sqrt{\mathbf{Q} \mathbf{H} \mathbf{Q}} \quad (22)$$

$$d \epsilon^p = d\lambda \frac{\partial F(\bar{Q})}{\partial \bar{Q}} = \frac{d\lambda}{\bar{Q}} \mathbf{H} \mathbf{Q}$$

where  $Q$  is the thermodynamic force associated to  $\epsilon^p$ .

This general form introduces three fourth order tensors  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{H}$  and one scalar quantity  $y$  to be identified by the homogenization process and from the microscopic quantities which are the material properties  $\mathbf{a}^\alpha$  and volume fraction  $c_\alpha$  of the  $n$  elastic phases ( $\alpha = 1..n$ ) and of the elastic-plastic phase ( $\mathbf{a}^\alpha = \mathbf{a}$ ,  $\mathbf{h}$ ,  $y$ ,  $c_0 = c$ , by dropping the index 0 relating to this phase).

For instance if elastic plastic decomposed homogenization is used (section 2.4) the macroscopic plastic strain (16) becomes:

$$\epsilon^p = c \mathbf{C}^T \epsilon^p \quad (23)$$

It should also be noted that using the stress localization tensor (17) for the elastic-plastic phase, the incompatibility plastic tensor reads:

$$\mathbf{i}_Q^p = \left( \frac{1}{c} \mathbf{k}_{QQ}^{-1} \mathbf{K}_{QQ} - \mathbf{I}_{QQ} \right) (\boldsymbol{\varepsilon}_Q^p - \boldsymbol{\varepsilon}_Q^p) \quad (24)$$

By substituting (24) in (17), the local stress  $\boldsymbol{\sigma}_Q$  is then given by

$$\boldsymbol{\sigma}_Q = \mathbf{c}_{QQ} \boldsymbol{\Sigma}_Q + \mathbf{c}_{LL} \boldsymbol{\Sigma}_L - \mathbf{k}_{QQ} \left( \frac{1}{c} \mathbf{k}_{QQ}^{-1} \mathbf{K}_{QQ} - \mathbf{I}_{QQ} \right) \boldsymbol{\varepsilon}^p \quad (25)$$

It should be noted that by using the localization tensor given in (13), the local stress could be formulated in a more usual form as a function of macroscopic tensors

$$\boldsymbol{\sigma} = \mathbf{C} \mathbf{Q}; \quad \mathbf{Q} = (\boldsymbol{\Sigma} - \mathbf{B} \boldsymbol{\varepsilon}^p) \quad \mathbf{B} = \begin{pmatrix} \frac{1}{c} \mathbf{K}_{QQ}^{-1} \mathbf{k}_{QQ} \mathbf{K}_{QQ} & -\mathbf{K}_{QQ} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (26)$$

which allows obtaining the local stress from macroscopic quantities. The equivalent stress  $\bar{\sigma}$  can be expressed as

$$\bar{\sigma}^2 = \mathbf{Q} \mathbf{H} \mathbf{Q}; \quad \mathbf{H} = \mathbf{c}^T \mathbf{h} \mathbf{c} \quad (27)$$

Substituting in local flow rule (2) and using (16) this provides the macroscopic plastic flow rule for  $\boldsymbol{\varepsilon}^p$

$$d \boldsymbol{\varepsilon}^p = d\lambda \frac{\partial F(\bar{Q})}{\partial \mathbf{Q}} = \frac{d\lambda}{Q} \mathbf{H} \mathbf{Q} \quad (\text{with } d\lambda = c d\lambda) \quad (28)$$

Hence, one achieves the determination of the three tensors  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{H}$  defining the anisotropic Prager's model.

### 3.2. Identification of the Prager's Model.

It remains to be proved that these homogenized constitutive equations and in particular the plastic flow rule can be identified with those of a Prager's model as presented in the beginning of the last section. The macroscopic elastic energy will be obtained from the averaging of its local counterparts, which can be written in mixed form as

$$\begin{aligned} w &= \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^e = \frac{1}{2} ((\boldsymbol{\varepsilon}_Q - \boldsymbol{\varepsilon}_Q^p) : \mathbf{k}_{QQ} (\boldsymbol{\varepsilon}_Q - \boldsymbol{\varepsilon}_Q^p) + \boldsymbol{\Sigma}_L : \mathbf{k}_{LL} \boldsymbol{\Sigma}_L) \\ w^\alpha &= \frac{1}{2} (\boldsymbol{\varepsilon}_Q : \mathbf{k}_{QQ}^\alpha \boldsymbol{\varepsilon}_Q + \boldsymbol{\Sigma}_L : \mathbf{k}_{LL}^\alpha \boldsymbol{\Sigma}_L) \\ \mathbf{W} = \langle w \rangle &= \frac{1}{2} (\boldsymbol{\varepsilon}_Q : \mathbf{K}_{QQ} \boldsymbol{\varepsilon}_Q + \boldsymbol{\Sigma}_L : \mathbf{K}_{LL} \boldsymbol{\Sigma}_L) - c \boldsymbol{\varepsilon}_Q^p : \mathbf{k}_{QQ} \boldsymbol{\varepsilon}_Q + \frac{1}{2} c \boldsymbol{\varepsilon}_Q^p : \mathbf{k}_{QQ} \boldsymbol{\varepsilon}_Q^p : \end{aligned} \quad (29)$$



By using (13) and (23), the last identity can easily be transformed into

$$\mathbf{W} = \frac{1}{2}(\epsilon_Q^e : \mathbf{K}_{QQ} \epsilon_Q^e + \Sigma_L : \mathbf{K}_{LL} \Sigma_L) + \frac{1}{2} \epsilon^p : \mathbf{B} \epsilon^p = \frac{1}{2} \epsilon^e : \mathbf{A} \epsilon^e + \frac{1}{2} \epsilon^p : \mathbf{B} \epsilon^p \quad (30)$$

with the elastic homogenized stiffness tensor  $\mathbf{A}$  is given by (12) and  $\mathbf{B}$  is given by (26). According to (20) the macroscopic thermodynamic force  $\mathbf{Q}$  associated to  $\epsilon^p$  is related to the local stress  $\sigma$  as in (26) by:

$$\sigma = \mathbf{C}\mathbf{Q}; \quad \mathbf{Q} = (\Sigma - \mathbf{B}\epsilon^p) \quad (31)$$

so that when expressed in terms of the macroscopic quantities, the equivalent stress  $\bar{\sigma}$ , is

$$\bar{\sigma}^2 = \bar{\mathbf{Q}}^2 \quad (32)$$

with

$$\bar{\mathbf{Q}}^2 = \mathbf{Q}\mathbf{H}\mathbf{Q}; \quad \mathbf{H} = \mathbf{c}^T \mathbf{h} \mathbf{c}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{QQ} & \mathbf{H}_{QL} \\ \mathbf{H}_{LQ} & \mathbf{H}_{LL} \end{bmatrix} \quad \left\{ \begin{array}{l} \mathbf{H}_{QQ} = \mathbf{H}_{QQ}^T = \mathbf{C}_{QQ}^T \mathbf{h}_{QQ} \mathbf{C}_{QQ} \\ \mathbf{H}_{LQ} = \mathbf{H}_{QL}^T = \mathbf{C}_{QL}^T \mathbf{h}_{QQ} \mathbf{C}_{QQ} + \mathbf{h}_{LQ} \mathbf{C}_{QQ} \\ \mathbf{H}_{LL} = \mathbf{H}_{LL}^T = \mathbf{C}_{QL}^T \mathbf{h}_{QQ} \mathbf{C}_{QL} + \mathbf{h}_{LQ} \mathbf{C}_{QL} + \mathbf{C}_{QL}^T \mathbf{h}_{QL} + \mathbf{h}_{LL} \end{array} \right.$$

In case of elastic homogeneity this simply reduce to  $\mathbf{H}=\mathbf{h}$ .

Substituting in the local flow rule (2) and using (23) and (26) this provides the macroscopic plastic flow rule for  $\epsilon^p$

$$d\epsilon^p = c \mathbf{C}^T d\epsilon^p = c \mathbf{C}^T d\lambda \frac{h\sigma}{\bar{\sigma}} = c d\lambda \frac{\mathbf{H}\mathbf{Q}}{\bar{\mathbf{Q}}} = d\Lambda \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} \quad (33)$$

with the macroscopic yield function

$$F(\mathbf{Q}) = \bar{\mathbf{Q}} - Y = f(\sigma), \quad \Lambda = c \lambda$$

This completes the identification of the homogenized constitutive equation with a Prager's model where  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{H}$  are respectively given by the elastic homogenization (12), (26) and (32).

In the special case of elastic homogeneity  $\mathbf{A}=\mathbf{a}$  and  $\mathbf{H}=\mathbf{h}$  so that the plastic heterogeneous behavior is contained in  $\mathbf{B}$ . In particular in case of isotropic phases the resulting anisotropy only appears in  $\mathbf{B}$ . this is however a very strong anisotropy as exemplified in [5].

#### 4. TWO PLASTIC POTENTIALS MODEL

Let us now consider the case where the investigated heterogeneous material consists in a mixture of two elastic plastic phases with any number of elastic phases. It can be expected in this case, that the homogenized process lead to a two internal variables model. As for the anisotropic Prager's model, this one is formulated in a more general form from an elastic energy

$$\begin{aligned}
 w(\epsilon^e, \epsilon^p, \Delta^p) &= \frac{1}{2} \epsilon^e : \mathbf{A} \epsilon^e + \frac{1}{2} \epsilon^p : \mathbf{B} \epsilon^p + \frac{1}{2} \Delta^p : \mathbf{Z} \Delta^p \\
 \Sigma &= \mathbf{A} \epsilon^e \quad \phi = \mathbf{Q} : d \epsilon^p + \mathbf{S} : d \Delta^p \geq 0 \\
 \mathbf{Q} &= (\Sigma - \mathbf{B} \epsilon^p) \quad \mathbf{S} = -\mathbf{Z} \Delta^p
 \end{aligned} \tag{34}$$

where  $\mathbf{Q}$  and  $\mathbf{S}$  are the thermodynamic forces associated to  $\epsilon^p$  and  $\Delta^p$  respectively. The giving of two yield functions  $F^\alpha(\mathbf{Q}, \mathbf{S})$ ,  $F^\beta(\mathbf{Q}, \mathbf{S})$  leads to the following flow rules

$$\begin{aligned}
 d \epsilon^p &= d \Lambda^\alpha \frac{\partial F^\alpha(\mathbf{Q}, \mathbf{S})}{\partial \mathbf{Q}} + d \Lambda^\beta \frac{\partial F^\beta(\mathbf{Q}, \mathbf{S})}{\partial \mathbf{Q}} \\
 d \Delta^p &= d \Lambda^\alpha \frac{\partial F^\alpha(\mathbf{Q}, \mathbf{S})}{\partial \mathbf{S}} + d \Lambda^\beta \frac{\partial F^\beta(\mathbf{Q}, \mathbf{S})}{\partial \mathbf{S}}
 \end{aligned} \tag{35}$$

In the decomposed homogenization case, and when only two material's phases (denoted by  $\alpha, \beta$ ) are elastic-plastic, the macroscopic plastic strain (16) is:

$$\epsilon_Q^p = \mathbf{K}_{QQ}^{-1} (c^\alpha \mathbf{k}_{QQ}^\alpha \epsilon_{QQ}^{p\alpha} + c^\beta \mathbf{k}_{QQ}^\beta \epsilon_{QQ}^{p\beta}) \tag{36}$$

which can be written by introducing the variable  $\Delta^p$  as:

$$\Delta^p = (\epsilon^{p\alpha} - \epsilon^{p\beta}) \tag{37}$$

in the two following forms:

$$\begin{aligned}
 \epsilon_Q^p &= \mathbf{K}_{QQ}^{-1} \mathbf{R}_{QQ} \epsilon_Q^{p\alpha} - c^\beta \mathbf{K}_{QQ}^{-1} \mathbf{R}_{QQ} \mathbf{k}_{QQ}^\beta \Delta^p \\
 \epsilon_Q^p &= \mathbf{K}_{QQ}^{-1} \mathbf{R}_{QQ} \epsilon_Q^{p\beta} + c^\alpha \mathbf{K}_{QQ}^{-1} \mathbf{R}_{QQ} \mathbf{k}_{QQ}^\alpha \Delta^p \quad \mathbf{R}_{QQ} = c^\alpha \mathbf{k}_{QQ}^\alpha + c^\beta \mathbf{k}_{QQ}^\beta
 \end{aligned} \tag{38}$$

By using (17) and (38) the local stresses in each of elastic plastic phases read:

$$\sigma^\alpha = c^\alpha \mathbf{Q} + \frac{1}{c^\alpha} \mathbf{S} \quad \sigma^\beta = c^\beta \mathbf{Q} - \frac{1}{c^\beta} \mathbf{S} \tag{39}$$

$$\mathbf{Q} = \Sigma - \mathbf{B} \epsilon^p, \quad \mathbf{S} = -\mathbf{Z} \Delta^p$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{K}_{\text{QQ}} \mathbf{R}_{\text{QQ}}^{-1} \mathbf{K}_{\text{QQ}} - \mathbf{K}_{\text{QQ}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} f^\alpha f^\beta \mathbf{k}_{\text{QQ}}^\alpha \mathbf{R}_{\text{QQ}}^{-1} \mathbf{k}_{\text{QQ}}^\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

One must remark that the constitutive tensor  $\mathbf{B}$  vanishes when all of the constituents are elastic plastic. To obtain the plastic flow rules for  $\epsilon^p$  and  $\Delta^p$ , the use of  $\epsilon^p$  and  $\Delta^p$  by the use of (36), (37), (38) and the following partial derivative properties

$$\begin{aligned} \frac{\partial F^\alpha(Q, S)}{\partial S} &= \frac{1}{c^\alpha} \frac{\partial F^\alpha(\sigma^\alpha)}{\partial \sigma^\alpha}, & \frac{\partial F^\beta(Q, S)}{\partial S} &= -\frac{1}{c^\beta} \frac{\partial F^\beta(\sigma^\beta)}{\partial \sigma^\beta} \\ \frac{\partial F^\alpha(Q, S)}{\partial Q} &= \mathbf{C}^{\alpha T} \frac{\partial F^\alpha(\sigma^\alpha)}{\partial \sigma^\alpha}, & \frac{\partial F^\beta(Q, S)}{\partial Q} &= \mathbf{C}^{\beta T} \frac{\partial F^\beta(\sigma^\beta)}{\partial \sigma^\beta} \end{aligned} \quad (40)$$

which gives:

$$\begin{aligned} d\epsilon^p &= d\lambda^\alpha \frac{\partial F^\alpha(Q, S)}{\partial Q} + d\lambda^\beta \frac{\partial F^\beta(Q, S)}{\partial Q} \\ d\Delta^p &= d\lambda^\alpha \frac{\partial F^\alpha(Q, S)}{\partial S} + d\lambda^\beta \frac{\partial F^\beta(Q, S)}{\partial S} \end{aligned} \quad (41)$$

$$\text{with } d\lambda^\alpha = c^\alpha d\lambda^\alpha \text{ and } d\lambda^\beta = c^\beta d\lambda^\beta$$

The two potentials model is completely determined for the case of decomposed homogenization by the tensors of fourth order  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{Z}$  and by two potentials defined by  $F^\alpha(Q, S)$  and  $F^\beta(Q, S)$ . By the same way as for the Prager's model (section 3.2.) and to prove that flow rules (41) can be identified to those presented in (20), one has to average the local elastic energy of each phase; this allow to obtain the macroscopic elastic energy. For the elastic phases the elastic energy reads:

$$w = \frac{1}{2} (\epsilon_Q : \mathbf{k}_{\text{QQ}} \epsilon_Q + \Sigma_L : \mathbf{k}_{\text{LL}} \Sigma_L) \quad (42)$$

For the plastic phases denoted by ( $\alpha$  and  $\beta$ ) one has:

$$\begin{aligned} w^\alpha &= \frac{1}{2} ((\epsilon_Q - \epsilon_Q^{p\alpha}) : \mathbf{k}_{\text{QQ}}^\alpha (\epsilon_Q - \epsilon_Q^{p\alpha}) + \Sigma_L : \mathbf{k}_{\text{LL}}^\alpha \Sigma_L) \\ w^\beta &= \frac{1}{2} ((\epsilon_Q - \epsilon_Q^{p\beta}) : \mathbf{k}_{\text{QQ}}^\beta (\epsilon_Q - \epsilon_Q^{p\beta}) + \Sigma_L : \mathbf{k}_{\text{LL}}^\beta \Sigma_L) \end{aligned} \quad (43)$$

The mean of these expressions gives by setting the same notations as in (34):

$$\begin{aligned}
w(\epsilon^e, \epsilon^p, \Delta^p) &= \frac{1}{2} \epsilon^e : \mathbf{A} \epsilon^e + \frac{1}{2} \epsilon^p : \mathbf{B} \epsilon^p + \frac{1}{2} \Delta^p : \mathbf{Z} \Delta^p \\
\Sigma &= \mathbf{A} \epsilon^e \quad \phi = \mathbf{Q} : d \epsilon^p + \mathbf{S} : d \Delta^p \geq 0 \\
\mathbf{Q} &= (\Sigma - \mathbf{B} \epsilon^p) \quad \mathbf{S} = -\mathbf{Z} \Delta^p
\end{aligned} \tag{44}$$

This identification is achieved by taking the plastic potential as the mean of the microscopic ones.

## 5. APPLICATIONS

To illustrate the obtained macroscopic two potentials model, let us consider the case of a fibrous composite. This heterogeneous medium consists in elastic plastic anisotropic fibre embedded in an anisotropic elastic plastic material. The fibers are aligned and are parallel and to  $\bar{x}_3$  direction (Fig.2.). The volume fractions of the fiber and the matrix are denoted respectively by  $c$  and  $(1-c)$ . Using classical notations, the elastic constitutive law read

$$\begin{pmatrix} \epsilon_{11} - \epsilon_{11}^p \\ \epsilon_{22} - \epsilon_{22}^p \\ \epsilon_{12} - \epsilon_{12}^p \\ \epsilon_{13} - \epsilon_{13}^p \\ \epsilon_{23} - \epsilon_{23}^p \\ \epsilon_{33} - \epsilon_{33}^p \end{pmatrix} = \begin{pmatrix} \frac{1}{E_T} & \frac{-\nu_T}{E_T} & 0 & 0 & 0 & \frac{-\nu_L}{E_L} \\ \frac{-\nu_T}{E_T} & \frac{1}{E_T} & 0 & 0 & 0 & \frac{-\nu_L}{E_L} \\ 0 & 0 & \frac{1+\nu_T}{E_T} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_L} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_L} & 0 \\ \frac{-\nu_L}{E_L} & \frac{-\nu_L}{E_L} & 0 & 0 & 0 & \frac{1}{E_L} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix} \tag{45}$$

where  $E_L$ ,  $\nu_L$ ,  $G_L$ ,  $E_T$  and  $\nu_T$  denote elastic properties of transverse isotropic material. The plastic behavior is defined by giving of plastic potential for each phase of this composite.

Following the result of section (4), the macroscopic mechanical behavior of this heterogeneous medium can be described by the so-called two plastic potentials model. This is defined by three constitutive tensors,  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{Z}$ .

As it has been shown, identification of all of these tensors is based on the vectorial decomposition of symmetric tensors (5). In the case of fibrous composite, symmetric tensors must be decomposed as in (8). Hence, the hybrid formulation (10) is obtained easily by inverting partially (45). So, one has

$$\begin{aligned}
 k_{QQ} &= E_L & k_{LQ} &= -k_{LQ}^T = -\begin{pmatrix} v_L & v_L & 0 & 0 & 0 \end{pmatrix} \\
 k_{LL} &= \begin{pmatrix} \frac{1}{E_T} - \frac{v_L^2}{E_L} & -\frac{v_T}{E_T} - \frac{v_L^2}{E_L} & 0 & 0 & 0 \\ -\frac{v_T}{E_T} - \frac{v_L^2}{E_L} & \frac{1}{E_T} - \frac{v_L^2}{E_L} & 0 & 0 & 0 \\ 0 & 0 & \frac{1+v_T}{E_T} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_L} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_L} \end{pmatrix}
 \end{aligned} \tag{46}$$

By setting  $\bar{E}_L, \bar{v}_L, \bar{G}_L, \bar{E}_T$  and  $\bar{v}_L$  as the effective elastic properties of the composite, some computations using eqns (11) and (12) gives easily:

$$\begin{aligned}
 \bar{E}_L &= \langle E_L \rangle & \bar{v}_L &= \langle v_L \rangle & \frac{1}{2\bar{G}_L} &= \langle \frac{1}{2G_L} \rangle & (47) \\
 \frac{1}{\bar{E}_T} &= \langle \frac{1}{E_T} \rangle - \langle \frac{v_L^2}{E_L} \rangle + \frac{\langle v_L \rangle^2}{\langle E_L \rangle} & \bar{v}_T &= \frac{\langle \frac{v_T}{E_T} \rangle + \frac{\langle v_L^2 \rangle}{\langle E_L \rangle} - \frac{\langle v_L \rangle^2}{\langle E_L \rangle}}{\langle \frac{1}{E_T} \rangle - \langle \frac{v_L^2}{E_L} \rangle + \frac{\langle v_L \rangle^2}{\langle E_L \rangle}}
 \end{aligned}$$

This mixture law obtained here for a two-phase composite is also available for more than two phases. As it is said in section 4 the tensor **B** vanishes (B=0) and the third constitutive tensor Z is obtained by replacing (46) in (39)

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c(1-c)E_L^m E_L^f}{cE_L^f + (1-c)E_L^m} \end{pmatrix} \tag{48}$$

where the superscripts m and f denote respectively matrix and fibre phases.

To complete a setting of the model one can take the plastic potential as the mean of the local ones. By the same way one can treat the case of the layered composite. If this structure consists in the same materials as before, i. e. two anisotropic elastic plastic phases denoted respectively by superscripts m and f, the effective elastic properties of this composite are

$$\begin{aligned}
 \frac{1}{2\bar{G}_L} &= \left\langle \frac{1}{2G_L} \right\rangle & \bar{\nu}_L &= \frac{\left\langle \frac{\nu_L E_T}{1 + \nu_T} \right\rangle}{\left\langle \frac{E_T}{1 + \nu_T} \right\rangle} \\
 \frac{1}{\bar{E}_L} &= \left\langle \frac{1}{E_L} \right\rangle - 2 \left\langle \frac{\nu_L^2 E_T}{E_L^2 (1 - \nu_T)} \right\rangle + 2 \frac{\left\langle \frac{\nu_L E_T}{1 + \nu_T} \right\rangle^2}{\left\langle \frac{E_T}{1 + \nu_T} \right\rangle^2} \left\langle \frac{E_T}{1 - \nu_T} \right\rangle & \bar{\nu}_T &= \frac{\left\langle \frac{\nu_T E_T}{1 - \nu_T^2} \right\rangle}{\left\langle \frac{E_T}{1 - \nu_T^2} \right\rangle} \\
 & & \bar{E}_T &= \frac{\left\langle \frac{E_T}{1 - \nu_T^2} \right\rangle^2 - \left\langle \frac{\nu_T E_T}{1 - \nu_T^2} \right\rangle^2}{\left\langle \frac{E_T}{1 - \nu_T^2} \right\rangle^2}
 \end{aligned} \tag{49}$$

Also the constitutive tensor B vanishes and the second tensor Z is given by:

$$Z = \frac{c(1-c)(1+\bar{\nu}_T)E_T^f E_T^m}{(1+\nu_T^m)(1+\nu_T^f)\bar{E}_T} \begin{pmatrix} \frac{(1-\bar{\nu}_T)}{(1-\nu_T^f)(1-\nu_T^m)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\bar{\nu}_T)}{(1-\nu_T^f)(1-\nu_T^m)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{50}$$

## 6. CONCLUSION

The analytical formulation purposed in this work allows the obtention of the macroscopic model in its explicit form. Till now, the effective response was derived by numerical calculations. By layered or decomposed homogenization, the analytical procedure, based upon the elastic energy, leads to the standard generalized materials formalism. The anisotropy of the macroscopic behavior is then characterized by the nature of the constitutive tensors. This nature is the manifestation of local interactions induced by elastic heterogeneity and plastic incompatibility between the constituents.

It seems that this formalism can also be obtained by different micromechanical procedures, whenever one considers that strain and stresses are uniform in every constituent of the heterogeneous medium.

It should also be emphasized that the main interesting features of this formulation, is that it can be implemented easily in finite element numerical code.

**ACKNOWLEDGEMENTS** - This work was supported by the Moroccan National Center of Coordination and Planning of Scientific and Technique Research (P.A.R.S., S.P.I. n°101).

## REFERENCES

1. Fougères R., Sidoroff F., The evolutive Masing model, Nuclear Engineering Design 114, 273, (1989).
2. Suquet P., Local and global aspects in the mathematical theory of plasticity, In: Sawczuk, Bianchi G., . (eds.), Plasticity Today., Elsevier, p. 279, (1984).
3. Suquet P., Element of homogenization for inelastic solid mechanics, In: Sanchez-Palencia, E., Zaoui, A. (eds.), Homogenization techniques for composite media. Lecture Notes in Physics. Springer-Verlag. Berlin. pp 193-278, (1987).
4. Suquet P., Effective properties on nonlinear composites CISM 377, Continuum Micromechanics, pp.197-264, (1997).
5. El Omri A., Sidoroff F., Homogenization of a two-phase elastic-plastic layered composite. C. R. Acad. Sc. Paris , 312, Série II, p .425, (1991).
6. El Omri A., Sidoroff F., Elastic plastic homogenisation for layered composite Eur. J. Mech. A/Solids 19, pp. 585 – 601, (2000).
7. Li J., Weng G.J., Time depenent creep of dual – phase viscoplastic mateial with lamella structure, Int. J. of Plasticity, Vol 14, N° 8, pp.755-770, (1998).
8. Dvorak G.J. and Teply J.L. On uniform Field in heterogeneous media, Proc. R, Soc, Lond. A, 431, 89,110, (1990).
9. Dvorak G.J., Bahei EL-Din A., Inelastic Composite materials: transformation analysis and experiments CISM 377, Continuum Micromechanics, 1-60, (1997).