THREE-DIMENSIONAL FINITE ELEMENT ANALYSIS OF ELASTOPLASTIC BEHAVIOR WITH FRICTIONAL CONTACT

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Abstract: In this paper, we present a theoretical and numerical analysis the incremental elastoplastic problems based on the finite element method and the mathematical programming. The survey is made on a perfect elastoplastic material obeying the criteria of von Mises. The contact with Coulomb’s dry friction to the tool-chip interface is taken into account and is formulated by the method of the implicit standard material (ISM), proposed by G. De saxcé and al, that led to only one variationnel principle of minimum in displacement for which contact and friction are coupled in the bipotential function non differentiable in certain points. In order to overcome this difficulty we suggest the use of the augmented Lagrangian method with Uzawa's algorithm. On the other hand the problem of coupling is solved by the use of an iteration procedure based on the fixed point method.

Keywords: elastoplasticity, mathematical programming, FEM, friction, bipotential, augmented Lagrangian method, fixed point method.

Introduction

That is to say homogeneous and isotropic structure Ω, subjected to a body force ∆F, imposed traction increments ∆τ on surface S, and the displacements imposed ∆τ on surface S. The part S = S ⊖ S ⊖ S of boundary such as S ⊖ S ⊖ S = φ is a zone suggested to be in contact with the structure Ω on the level of tangent common plain π as indicates the following figure:

![Figure 1. Structure 3D with boundary conditions](image)

The relative velocity ̇u and the contact traction t can be decomposed in their normal and tangential components as follows:

\[ ̇u = ̇u_n + ̇u_t, \quad t = t_n + t_t \]

where n is the outward unit normal to domain Ω and ̇u and ̇u are, respectively, normal and tangential components of the relative velocity ̇u; t and t indicate normal and tangential components of the contact traction t.

The complete unilateral contact law with Coulomb’s friction can be written in the following form:

\[
\begin{align*}
\text{If} \quad t_n &= 0 \quad \text{then} \quad ̇u_n \geq 0 \quad \text{! no contact} \\
\text{Else if} \quad t_n > 0 \quad \text{and} \quad ∥n∥ < μt_n \quad \text{then} \quad ̇u_n = 0 \quad \text{! sticking} \\
\text{Else} \quad (t_n > 0 \quad \text{and} \quad ∥n∥ = μt_n) \quad \text{then} \quad ̇u_n = 0 \quad \text{and} \quad \exists ̇λ \geq 0 \\
\text{such that} \quad -̇u_t = ̇λt_t \quad \text{! sliding}
\end{align*}
\]

where μ indicates the coefficient of friction.

The inverse law can be described as

\[
\begin{align*}
\text{If} \quad ̇u_n = 0 \quad \text{then} \quad t = 0 \quad \text{! no contact} \\
\text{Else if} \quad ̇u_n = 0 \quad \text{and} \quad ∥n∥ < μt_n \quad \text{! sticking} \\
\text{Else} \quad ̇u_n \neq 0, \quad ̇u_n = 0 \quad \text{and} \quad t = -μt_n ̇u_t / ∥n∥ \quad \text{! sliding}
\end{align*}
\]

Let \( K_μ \) denote Coulomb’s cone of friction:

\[ K_μ = \{ (t_n, t_t) \quad \text{such that} \quad f(t_n, t_t) = |n - μt_n| \leq 0 \} \]

This law of contact doesn’t admit a surpotential but she admits a biconvex bipotential \( b_μ(-̇u, t) \) [1]:

\[ b_μ(-̇u, t) = \begin{cases} 
μn ̇u - ̇u_n & \text{if} \quad t \in K_μ \quad \text{and} \quad ̇u_n \geq 0 \\
+∞ & \text{otherwise} \end{cases} \]

Therefore, the complete contact law with friction becomes

\[ -̇u \in ∂_+ b_μ(-̇u, t) \quad \text{et} \quad t \in ∂_+ b_μ(-̇u, t) \]

1. Elastoplastic evolution with frictional contact

2.1. Elastoplastic analysis

The total strain increment can be decomposed into elastic and plastic parts:

\[ Δε = Δε^e + Δε^p \]

where \( Δε^e \) is the elastic strain increment defined by the Hooke’s law and \( Δε^p \) is the plastic strain increment.

Let us consider the following incremental notations:

\[ Δτ = τ_n - τ, \quad Δσ = σ_n - σ, \quad Δε = ε_n - ε, \quad Δε^p = Δσ/Δε^p \]

where the index 0 (resp. 1) is relative to beginning (resp. to the end) of the step, \( Δσ/Δε^p \) is the plastic strain rate given by the normality law \( (ε^p = λ/μ, \quad f \) is the yield function and \( λ \) is plastic multiplier).

We use the concept of the inf-convolution to calculate the incremental elastoplastic supertension \( ΔV(Δε) \):

\[ ΔV(Δε) = (ΔV_e ⊖ ΔV_p)(Δε) \]

where

\[ V_e = \inf_{Δε^p, \text{admissible}} (ΔV_e(Δε - Δε^p) + ΔV_p(Δε^p)) \]

We obtain finally the incremental elastoplastic supertension in term of strain for a material obeying the von-Mises criterion by the following algorithm:

\[
\begin{align*}
\text{If} \quad Δε^e &\geq \frac{σ_n}{G√6} \quad \text{then} \\
ΔV(Δε) &= \frac{1}{2} K_e (Δε_n)^2 + G∥Δε∥^2 \\
\text{Else} \quad Δε^e &= 0 \quad \text{and} \quad ΔV(Δε) = \frac{1}{2} K_e (Δε_n)^2 + G∥Δε∥^2
\end{align*}
\]
where $\| \|$ denotes the Euclidean norm, $K_e$ is the factor of compressibility, $G$ is the modulus of rigidity of Coulomb, $\sigma$ is the yield stress of material considered, $\Delta e$ and $\Delta u$ are respectively deviatoric and spherical parts of the tensor of elastoplastic strain $\Delta \varepsilon : [\Delta \varepsilon]^e$ is the Euclidean norm of the plastic strain deviator.

Finally, the incremental law of material is defined by:

$$
\begin{align*}
\Delta \sigma & \in \partial_{\sigma} \Delta V(\Delta \varepsilon) \\
\Delta \varepsilon & \in \partial_{\varepsilon} \Delta W(\Delta \sigma)
\end{align*}
$$

where $\Delta W(\Delta \sigma)$ is the dual incremental elastoplastic surpotential $\Delta V(\Delta \varepsilon)$ is the Euclidean formulation of a mechanical problem with friction.

2.2. Frictional contact analysis

The solution of the irregular contact law with Coulomb’s dry friction consists to express it in incremental terms and for this reason one defines the whole of the admissible increments $\Delta K^p$, of traction by:

$$
\Delta K^p = \{(\Delta t, \Delta \tau) \text{ such that } [\Delta t + \Delta \tau] \leq [\mu(t, \Delta \tau)] \}
$$

The incremental formulation of the contact law with Coulomb’s dry friction is expressed by the incremental bipotential proposed by De Saxce and Feng [1998]:

$$
\Delta t(\Delta u, \Delta \tau) = t_0(\Delta u) + t_1(\Delta u, \Delta \tau) + \mu(t_0 + \Delta t, \Delta \tau) \left[ \Delta u \right]
$$

where the two solids are initially spaced $h_0$.

The corresponding incremental contact laws take the form

$$
\begin{align*}
\Delta \tau & \in \partial_{\tau} \Delta b \left( -\Delta u, \Delta \tau \right) \\
-\Delta u & \in \partial_{\varepsilon} \Delta b \left( -\Delta u, \Delta \tau \right)
\end{align*}
$$

We will present in the continuation the variational formulation of a mechanical problem with friction.

2. Variational formulation

Let us consider the structure with the boundary conditions shown on figure 1.

A field of the increments of displacements is known as kinematically acceptable (KA) if the following equations of compatibility are satisfied:

$$
\begin{align*}
\Delta \varepsilon(u) &= \text{grad}_u \Delta u^k \text{ in } \Omega \\
\Delta u^k &= \Delta \tau \text{ on } S_i
\end{align*}
$$

A field of the increments of constraints is known as statically acceptable (SA) if the following equilibrium equations are satisfied:

$$
\begin{align*}
\text{div}_u (\Delta \varepsilon^u) + \nabla \tau &= 0 \text{ in } \Omega \\
\Delta u(\Delta \varepsilon^u) &= \Delta \sigma^u \text{ on } S_i
\end{align*}
$$

We obtain the following function, called bifunctional starting from the principle of the virtual power.

$$
\Delta \Phi(\Delta u, \Delta \sigma) = \int_{\Omega} \Delta V(\Delta \varepsilon(u)) - \Delta \sigma \cdot \Delta u \frac{d\Omega}{\partial} + \int_{S_i} \Delta b^k \left( -\Delta u, \Delta \tau(\Delta \sigma) \right)dS
$$

Therefore, the kinematically variational principle becomes:

$$
\inf_{\Delta u^k, \Delta \sigma^k} \Delta \Phi(\Delta u^k, \Delta \sigma^k)
$$

3. Finite elements discretization

The displacement and strain fields are expressed with respect to an unknown nodal displacement vector $\Delta U$ as

$$
\begin{align*}
\Delta u(x) &= N(x)^{} \Delta U \\
\Delta \varepsilon &= B(\Delta U) \text{ with } B = \nabla_i (N(x))
\end{align*}
$$

where $x = (x, y, z)$ are the nodal coordinates, $N(x)$ is the matrix of the shape functions and $\nabla_i$ is the symmetric gradient operator.

Let us introduce the generalized nodal force increment vector

$$
\Delta F = \int_{\Omega} N^T \Delta \tau d\Omega + \int_{S} N^T \Delta \tau dS
$$

The bifunctional (17) takes the following discretized form:

$$
\Delta \Phi(\Delta U) = \int_{\Omega} \Delta V(\Delta U) d\Omega - \Delta F + \int_{S} \Delta b(\Delta U) dS
$$

In this case, the increments of stresses aren’t discretized like the principal stresses, but can be deduced starting from the value from the increments from displacements by the equations:

$$
\begin{align*}
\int_{\Omega} B^T \Delta \sigma d\Omega - \Delta F - \int_{S} N^T \Delta \tau dS &= 0 \\
\Delta \sigma &= \frac{\partial V(B(\Delta U))}{\partial \Delta u}
\end{align*}
$$

The problem of coupling of the traction increments with those of displacements is solved by using an iterative procedure based on the fixed point method. The integrals are calculated numerically by the Gauss method.

4. Augmented Lagrangian method with Uzawa’s algorithm

The bipotential of the contact with friction isn’t differentiable everywhere what poses problems on the level of the mathematical programming. In the case of the contact, we obtain the following equation:

$$
\Delta \tau = \text{proj} \left( \Delta \tau - \rho \left( \Delta u + \Delta u + h_0 + \rho \left[ \Delta u \right] \right) \right)
$$

where $\rho$ is a positive coefficient whose value must be quite selected to ensure numerical convergence.

The application of the Uzawa’s algorithm of projection leads to an iterative diagram made up of two stages:

**Prediction**:

$$
\begin{align*}
\Delta \tau^* &= \Delta \tau - \rho \left( \Delta \tau(\Delta \varepsilon^u) + \Delta u + \rho \left[ \Delta \tau(\Delta \sigma^u) \right] \right) \\
\Delta \sigma^* &= \Delta \tau - \rho \Delta \tau(\Delta \sigma^u)
\end{align*}
$$

**Correction**:

$$
\Delta \tau^{i+1} = \text{proj} \left( \Delta \tau^{i}, \Delta K^p \right)
$$

where $\Delta \tau^{i+1}$ is the prediction of the contact traction to iteration $i+1$.

5. Fixed point method

The couple $(-\Delta u, \Delta \sigma)$ , solution of the problem in extreme cases, is with the intersection of the hyper-surface ($\Gamma$) defined by the constitutive laws (22b, c) and of under space closely connected (EA) of the solutions kinematically admissible, defined by the equation (22a).
Figure 2. Diagram of the fixed point method

The resolution of (22b, c) corresponds at the local stage, represented by the direction of assembled $(E^+)$, and the resolution of (22a) corresponds at the total stage, represented by the direction of descent $(E^-)$.

5.1. Total stage

$\Delta \sigma$ is fixed and a new approximation of $\Delta u^{t+1}$ is calculated by the minimization of (22a) using the code of programming maths Minos [3]. The problem of displacements consists in minimizing the equation (21).

5.2. Local stage

$\Delta U^{t+1}$ is fixed and a new approximation of the increment of the reaction $\Delta t^{t+1}$ is calculated:

$$
\Delta t^{t+1} = \frac{\partial \Delta u^{t+1}}{\partial (\Delta u)}
$$

The criterion of stop of the iterations is based on the error relating to the implicit constitutive laws. The estimate of following error is considered:

$$
\varepsilon_{\text{err}} = \frac{\int \| \Delta \sigma - \Delta \sigma \| d\Omega + \int \| \Delta u - \Delta t \| d\Omega}{\sqrt{\int \| \Delta \sigma \| d\Omega + \int \| \Delta t \| d\Omega}} \leq \varepsilon_{\text{tol}}
$$

with $\varepsilon_{\text{tol}}$ is the value of tolerance which depends on the sought precision and the cost of calculation.

6. Numerical application (Punching of a thick plate)

We consider the frictional contact between a thick plate and a rigid punch. The parameters for this problem are: yield stress $\sigma_y = 300$ MPa, Young’s modulus $E = 210000$ MPa, Poisson’s ratio $v = 0.3$, Friction coefficient $\mu = 0.1$ and the dimensions of the punch are $x \times y \times z = 8 \times 8 \times 4$.

Due to symmetry of geometry, only one quarter of the workpiece is modelled.

References