NON-LINEAR NORMAL MODES OF BEAMS ON ELASTIC FOUNDATIONS

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Introduction

Thin beams are widely used in many civil, mechanical and aerospace engineering applications. Also, the problem of beams resting on non-linear elastic foundation is very often encountered in the analysis of building, highway, and railroad structures. The knowledge of the non-linear normal modes and non-linear frequencies of such structures constitutes an important part in the dynamic analysis under loading conditions.

The present study focuses on the non-liner free vibration of simply supported-simply supported (SS) beams resting on an elastic foundation using a multimodal approach. The transverse displacement is expanded as a sum of products of unknown time functions and the linear modes of free vibration of the SS beam. The equations of motion in the time domain are obtained by applying Lagrange's equations. The unknown time functions are expressed as a Fourier cosine series and the harmonic balance method is utilized to derive the equations of motion in the frequency domain. These non-linear algebraic equations represent a non-linear eigenvalue problem, which is solved by iterative numerical methods, namely the Linearized Updated Mode (LUM) method and the arc-length continuation method. The one harmonic solution is compared with available numerical results. Also, an internal resonance of order 1:3 is discovered and its consequences discussed.

Mathematical Formulation

Consider the transverse vibrations of a slender uniform beam resting on a non-linear Winkler elastic foundation. The characteristics of the beam are: L, length of the beam; S, area of the cross-section; I, moment of inertia; E, Young's modulus and r, the mass per unit volume. For such a beam, the total potential energy V is the sum of the strain energy due to the bending and the potential energy due to the Winkler-type elastic foundation of parameters a_1 and b_1:

$$V = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 W}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^L \left(a_1 W + \frac{1}{2} b_1 W^3\right) W \mathrm{d}x \quad (1)$$

The kinetic energy of the beam is given by

$$T = \frac{rS}{2}\int_0^L \left(\frac{\partial^2 W}{\partial t^2}\right)^2 \mathrm{d}x \quad (2)$$

Using a generalized parameterization and the usual summation convention, the transverse displacement $W(x,t)$ can be expanded in the form of the following finite series

$$W(x,t) = q_i(t)\, w_i(x) \qquad i = 1-n \quad (3)$$

where w_i are the transverse basic functions taken as the linear modes of free vibration of the beam depending on the boundary conditions considered. These beam functions are given explicitly in Ref. [1] in the case (SS) and clamped-clamped (CC) beams. q_i are the corresponding generalized parameters and n is the number of the basic functions used in the model. Substituting W in the expressions of the total potential energy V and the kinetic energy T, and rearranging, leads to

$$V = \frac{1}{2} q_i q_j k_{ij} + \frac{1}{4} q_i q_j q_k q_l b_{ijkl} \quad (4)$$

$$T = \frac{1}{2} \dot{q}_i \dot{q}_j m_{ij} \quad (5)$$

In the above expressions m_{ij}, k_{ij} and b_{ijkl} are, respectively, the general terms of the mass, the linear rigidity and the fourth order non-linearity foundation tensors. These tensors are given by

$$k_{ij} = EI\int_0^L \frac{\mathrm{d}^2 w_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2 w_j}{\mathrm{d}x^2}\mathrm{d}x + a_1\int_0^L w_i w_j \mathrm{d}x$$
$$b_{ijkl} = b_1\int_0^L w_i w_j w_k w_l \mathrm{d}x, \quad m_{ij} = rS\int_0^L w_i w_j \mathrm{d}x \quad (6)$$

The governing equations of motion derived from Lagrange's equations are given by

$$\ddot{q}_i m_{ir} + q_i k_{ir} + q_i q_j q_k b_{ijkr} = 0, \quad r = 1-n \quad (7)$$

Non-dimensional formulation is introduced by putting

$$x^* = \frac{x}{L}, \; w_i(x) = L w_i^*(x^*), \; t = t\left(\frac{rSL^4}{EI}\right)^{1/2} \quad (8)$$

Equations (7) can be written in non-dimensional form as

$$\ddot{q}_i m_{ir}^* + q_i k_{ir}^* + q_i q_j q_k b_{ijkr}^* = 0, \quad r = 1-n \quad (9)$$

The m_{ij}^*, k_{ij}^* and b_{ijkl}^* terms are non-dimensional tensors related to the dimensional ones by the following equations:

$$m_{ij} = \rho SLh^2 m_{ij}^*,\ (k_{ij}, b_{ijkl}) = \frac{EIh^2}{L^3}(k_{ij}^*, b_{ijkl}^*) \qquad (10)$$

These tensors are given by

$$k_{ij}^* = \int_0^1 \frac{d^2 w_i^*}{dx^{*2}} \frac{d^2 w_j^*}{dx^{*2}} dx + \alpha \int_0^1 w_i^* w_j^* dx^*$$
$$b_{ijkl} = \beta \int_0^L w_i^* w_j^* w_k^* w_l^* dx^*,\quad m_{ij} = \int_0^1 w_i^* w_j^* dx^* \qquad (11)$$

where $\alpha = \frac{\alpha_1 L^4}{EI}$ and $\beta = \frac{\beta_1 L^6}{EI}$ are the non-dimensional parameters of the elastic foundation.

Equation (9) can be written in matrix form as

$$\left[M^*\right]\{\ddot{q}\} + \left[K^*\right]\{q\} + \left[K_{nl}^*(\{q\})\right]\{q\} = 0 \qquad (12)$$

where $\left[M^*\right]$ and $\left[K^*\right]$ are the mass and linear rigidity bending matrices, respectively. Each term of the non-linear foundation matrix $\left[K_{nl}^*\right]$ is a quadratic function of the column matrix of the transverse generalized parameters $\{q\} = \left[q_1\ q_2 \cdots q_n\right]^T$. Eq. (12) represents a set of coupled Duffing's equations for which an exact mathematical solution can be obtained only in the one-dimensional case corresponding to the single-mode approximation, in terms of elliptic functions. In the multidimensional case, approximate solutions may be obtained by perturbation methods or the harmonic balance method. Assuming periodic (not purely harmonic) motions of the beam, the vector of transverse generalized parameters may be expressed as

$$\{q(t)\} = \sum_{j=1}^{k} \left\{A^{(2j-1)}\right\} \cos((2j-1)\omega^* t) \qquad (13)$$

in which only odd harmonics are retained, due to the character of cubic non-linearity of the system. The influence of the first three harmonics on the response of the beam resting on non-linear elastic foundation will be investigated in the present work. Inserting Eq. (13), with $k = 3$, into Eq. (12) and applying the harmonic balance method, by neglecting harmonics higher than $5\omega^* t$, the following set of non-linear algebraic equations is thus obtained

$$\left(\left[KL^*\right] + \left[KNL^*(\{A\})\right]\right)\{A\} = \omega^{*2}\left[M^{1*}\right]\{A\} \qquad (14)$$

The mass matrix $\left[M^{1*}\right]$ is a diagonal block matrix in which the diagonal blocks are respectively $\left[M^*\right]$, $9\left[M^*\right]$ and $25\left[M^*\right]$. The linear stiffness matrix $\left[KL^*\right]$ is also a diagonal block matrix having the same diagonal block $\left[K^*\right]$. The non-linear foundation matrix $\left[KNL^*\right]$, which is a function of the fourth order non-linearity tensor b_{ijkl}^* and depends quadratically on the vector of contribution coefficients corresponding to each harmonic $\{A\} = \left[A^1 \quad A^3 \quad A^5\right]^T$, is given in detail in Ref. [2]. The set of non-linear algebraic equations (14) represents a non-linear eigenvalue problem, which can be solved iteratively by the so-called linearized updated mode (LUM) method in the case of simple harmonic motion. This method is based on a consecutive solution of an eigenvalue problem [2, 3]. However, in the multi-harmonic vibration problem, convergence is difficult with the LUM method as pointed out in Ref. [2]. Therefore, a continuation method using the arc-length as a parameter is utilized to obtain the multi-harmonic response. The parameter of the continuation method is obtained by constraining the distance between the two successive points of the backbone curve to a fixed value. This method is presented in detail in Ref. [2].

Results and Discussion

In this work, only the fundamental non-linear normal mode of a SS beam resting on non-linear elastic foundation (with $\alpha = 0$ and $\beta = 10^6$) is investigated. In the case of simple harmonic motion assumption, the convergence study with the number of basic functions n to be used in the model is made by using the LUM method, in which the parameter is the amplitude of vibration, rather than by the continuation method, in which the parameter is the arc-length. In fact, the two methods give the same results in the case of one harmonic approximation.

The numerical results of the non-linear frequency ratio ω_{nl}/ω_l, obtained for a maximum non-dimensional amplitude of $w_{max}^* = 0.063$ at the middle of the beam, are summarized in Table 1. It can be seen that a good accuracy is achieved by using the first five symmetric SS beam functions ($n = 5$).

Table 1: Convergence of the frequency ratio ω_{nl}/ω_l of free vibrations of a SS beam on elastic foundation with the number of basic functions n for $w_{max}^* = 0.063$

n	1	2	3	4	5	6
ω_{nl}/ω_l	4.8907	5.1355	5.1181	5.1188	5.1187	5.1187

In Table 2, the SS beam fundamental frequency ratio ω_{nl}/ω_l computed with one harmonic, for various values of the maximum amplitude of vibration w_{max}^*, is presented and compared with that of the invariant manifold approach [4] and the power series method [5]. Good agreement is noticed between the three methods.

Although the present one harmonic solution is quite accurate compared with the published literature, the influence of higher harmonics must be investigated. It can be seen from Figure 1, in which the backbone curves of the SS beam on non-linear elastic foundation obtained with one, two and three harmonics, that the one harmonic approximation is only accurate for maximum non-dimensional amplitudes of about 0.0343. Examination of Figure 1 shows also that the two harmonics solution is very accurate.

Table 2: Frequency ratio w_{nl}/w_l of free vibrations of a SS beam on elastic foundation for $w^*_{\max} = 0.063$

$w^*_{\max}$	Ref. [4]	Present	Ref. [5]
0.01666	1.6133	1.6167	1.7028
0.0333	2.7209	2.7530	2.9775
0.05	3.9289	4.0427	4.3415
0.063	4.8907	5.1187	5.3341

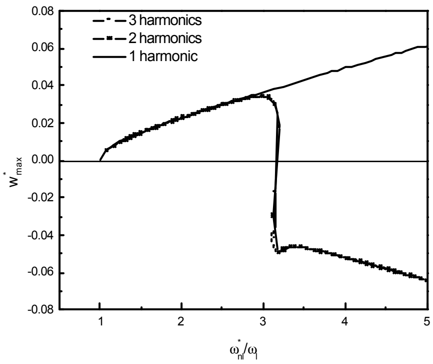

Fig. 1. Non-dimensional total amplitude at the middle of the SS beam versus the non-linear frequency ratio.

The first two non-dimensional linear frequencies of the SS beam are $w_{l1} = p^2$ and $w_{l2} = 9p^2$ (for $a = 0$). When the non-dimensional non-linear fundamental frequency w_{nl1} increases, the second resonance frequency can be equal to three times the first one. Therefore, a 1:3 internal resonance may occur due to coupling between the symmetric modes 1 and 2. In what follows two harmonic terms will be considered, which is sufficient in the case of this type of internal resonance. The backbone curves, obtained at the point $x^*=1/2$, corresponding to the first and the third harmonics are given in Figures 2(a) and 2(b). The dashed curves are merely the symmetrical of the solid ones with respect to the w_{nl}/w_l –axis. In Figures 3(a) and 3(b) the mode shapes associated with the first and the third harmonics at various maximum non-dimensional amplitudes are shown. It appears from these figures that the first harmonic vibrates at the first symmetric mode and that the third harmonic is increasingly affected by the second symmetric mode than by the first.

The backbone curve of the first harmonic shows a hardening type non-linearity at the beginning. However, when the non-dimensional non-linear fundamental frequency is increased in the vicinity of $3w_{l1}$, the third harmonic amplitude begins to grow: a 1:3 internal resonance occurs. As the amount of energy that is transferred from the first mode to the second mode increases, the amplitude of the first harmonic begins to decrease and quickly drops vertically to cross the w_{nl}/w_l –axis around 3.132. At this frequency, the importance of the first harmonic begins to increase (in absolute value) again and the third harmonic backbone curve forms a loop and decreases due to getting out of the internal resonance.

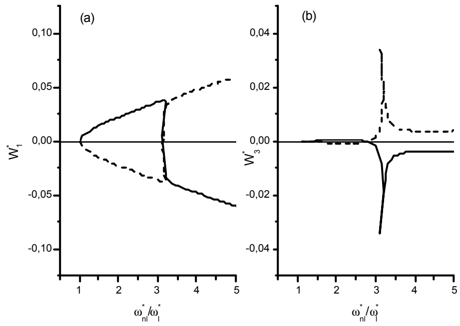

Fig. 2. Backbone curves of the SS beam on elastic foundation: (a) first harmonic, (b) third harmonic.

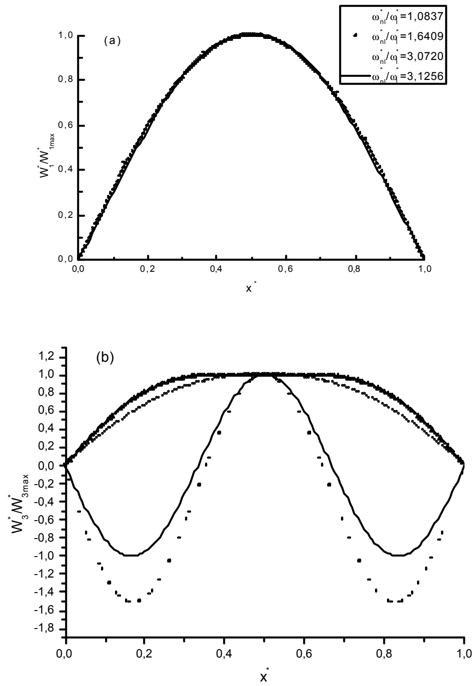

Fig. 3. Mode shapes of the SS beam on elastic foundation: (a) first harmonic, (b) third harmonic.

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