NONLINEAR DYNAMIC OF A CIRCULAR RING NEAR THE PRINCIPAL RESONANCE

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Introduction

Several structures having axisymmetric geometry can be modeled by circular rings which exhibit most of the dynamic characteristics of such structures. The most important theoretical and experimental work on the nonlinear vibration of circular rings is due to Evensen [1], Dowell and Ventres [2]. In this work we examine the nonlinear flexural vibrations of free non-rotating thin circular ring using Sander's approach. In the first part, a multimode approach formulation based on Lagrange's equation is developed and new formulation is provided. This formulation follows the nonlinear Sander's theory that consider the same expression for the radial and circumferential displacements as those used in [3, 4]. For testing and comparing the proposed model, we analyze nonlinear free responses in the case of a SDOF model. In the second part of this work, we examine the dynamic of the external forced considered mode of the ring. Using the multiple scales technique, the slow flow is derived and then the trivial and non trivial periodic responses of the mode are obtained in the vicinity of the principal resonance 1:1. Bifurcation of these solutions are investigated as well.

Equations of motion

Consider a circular ring of radius R and thickness h. It is assumed that the ring is thin, and its materiel is linearly elastic, homogeneous and isotropic. As for shallow cylindrical shell, equations of motion for thin rings can be deduced by considering the following assumptions [1]:

- The radial displacement W, the tangential displacement V, and the applied load q are taken to be functions of only two variables, the circumferential coordinate y and time t.
- The ring thickness h and its width are both taken to be constant. The ring is assumed to be thin, such that $(h/R)^2 << 1$.
- The longitudinal in-plane force N_x and the shearing force N_{xy} are both taken to be zero throughout the ring.

Radial and circumferential displacements

The radial and circumferential displacements, including the driven and companion modes, are chosen in the single mode approach to the n^{th} mode adopted by Evensen [1] and Natsiavas [4, 5]

Total kinetic and strain energies

The total strain energy V_T of the circular thin ring can be written as the sum of the strain energy due to bending, denoted as V_b, plus the strain energy due to the membrane induced by large deflexion V_m. Then, $V_T = V_b + V_m$. Under the classical ring bending theory, V_m is neglected, V_b and the kinetic energy are expressed as

$$V_b = \frac{D}{2}\int_0^{2\pi R}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 dy \qquad (1)$$
$$T = \frac{\rho h}{2}\int_0^{2\pi R}\left[\left(\frac{\partial W}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right]dy$$

were D is bending stiffness defined by $\frac{Eh^3}{12(1-\nu^2)}$

E is Young's modulus, υ is Poisson's ratio and ρ, h are respectively the mass density and the ring thickness. In order to validate the present multimode model, we start by assuming a single mode response and comparing the obtained results with the theoretical and experimental ones obtained by Evensen [1].

If only one mode is assumed, and in the case without in-plane inertia, equation of motion takes the form

$$\ddot{q} + bq(\ddot{q}q + \dot{q}^2) + kq = 0 \qquad (2)$$

Free vibration

Assuming a harmonic response q(t)=Asin(wt), applying the harmonic balance method, we obtain the following equation

$$\frac{\omega_{nl}^2}{\omega_l^2} = \frac{1}{1 + \frac{\varepsilon(1-n^{-2})^4}{4}\bar{A}^2} \qquad (3)$$

where $\bar{A} = \frac{A}{h}$, ω_{nl} is the nonlinear frequency, ω_l is the linear frequency given by $\omega_l^2 = \frac{k}{m}$, k and m are given for the n^{th} mode by:

$$k = \frac{Dn^4\pi}{R^3},\ m = \rho hR\pi,\ \frac{b}{2m} = \frac{(1-n^{-2})^4 n^4}{4R^2} \text{ and } \varepsilon = \left(\frac{n^2 h}{R}\right)^2$$

Fig. 1 illustrates comparison between the backbone curves given by Evensen [1] in his first formulation and that obtained here when Sander's model is considered with and without in-plane inertia for a circular thin ring. It is shown that a softening behaviour is accomplished.

Periodic solutions and bifurcation

In this section we examine periodic responses of the considered fourth mode given by Eq. (2). If we introduce a periodic external forcing and a weak damping, Eq. (2) can be written as

$$\ddot{\tilde{q}} + b\tilde{q}(\ddot{\tilde{q}}q + \dot{\tilde{q}}^2) + c\dot{\tilde{q}} + k\tilde{q} = F\cos(\tilde{\Omega}t) \tag{4}$$

where $\tilde{q} = hq, \mu = \dfrac{c}{\omega^2 mh}, \tau = \omega t, \quad \Omega = \dfrac{\tilde{\Omega}}{\omega}$

$\dfrac{b}{m} = 1/2\beta\varepsilon,\ \beta = (1-n^{-2})^4,\ \varepsilon = (\dfrac{h}{R}n^2),\ G = \dfrac{F}{\omega^2 mh}$

Then Eq. (4) can be written in the non dimensional form as

$$\ddot{q} + \frac{1}{2}\beta\varepsilon q(\ddot{q}q + \dot{q}^2) + \mu\dot{q} + q = G\cos(\Omega\tau) \tag{5}$$

To construct an approximation of periodic solutions to Eq. (5), we scale amplitude of forcing and damping parameters as $G = \varepsilon\tilde{G}, \mu = \varepsilon\tilde{\mu}$.

We perform our analysis of periodic motions in the vicinity of the principal resonance 1:1 which is expressed as $\tilde{\Omega} = 1 + \varepsilon\sigma$ where σ is a detuning parameter. Using the method of multiple scales [6], we construct a uniform approximation of the solution to Eq. (5) in the form

$$q(\tau) = q_0(T_0, T_1) + \varepsilon q_1(T_0, T_1) \tag{6}$$

where $T_0 = \tau$ is a fast time scale and $T_1 = \varepsilon\tau$ is a slow time scale describing the envelope of the response. In terms of the variable T_n the time derivative becomes $\dfrac{d}{d\tau} = D_0 + \varepsilon D_1 + o(\varepsilon^2)$ where $D_n = \dfrac{\partial}{\partial T_n}$.

The solution of Eq. (5) in different order of ε can be expressed as

$$\begin{aligned} q_0(T_0, T_1) &= A(T_1)e^{iT_0} + cc \\ q_1(T_0, T_1) &= \frac{\beta A(T_1)}{8}e^{3iT_0} + cc \end{aligned} \tag{7}$$

where cc denotes the complex conjugate of the preceding terms. The quantity $A(T_1)$ is to be determined by eliminating the secular terms at the next level of approximations.

Letting $A(T_1) = \dfrac{1}{2}a(T_1)e^{\theta(T_1)}$ where a and θ are real functions, separating real and imaginary parts, we obtain the following slow flow modulation equations of amplitude and phase

$$\begin{aligned} \frac{da}{dT_1} &= -\frac{\tilde{\mu}a}{2} + \frac{1}{2}\tilde{G}\sin(\gamma) \\ a\frac{d\gamma}{dT_1} &= a\sigma + \frac{\beta a^3}{8} + \frac{1}{2}\tilde{G}\cos(\gamma) \end{aligned} \tag{8}$$

where $\gamma = \sigma T_1 - \theta$.

The stationary regimes (fixed points) of the slow flow (8) corresponding to periodic solutions of the original Eq. (5) are given by setting the time derivatives equal to zero, $\dot{a} = 0$ and $\dot{\gamma} = 0$. Eliminating γ from the algebraic system, we obtain

$$(\frac{\tilde{\mu}a}{2})^2 + (a\sigma + \frac{\beta a^3}{8})^2 = \frac{\tilde{G}^2}{4} \tag{9}$$

This last equation can be transformed to the cubic one

$$AX^3 + BX^2 + CX + D = 0 \tag{10}$$

with

$A = (\beta/8)^2,\ B = (\beta\sigma/4), C = \sigma^2 + (\tilde{\mu}/2)^2, D = -\dfrac{\tilde{G}^2}{4}$

and $X = a^2$.

By writing $X = R - (B/3A)$ and X= R-(B/3A) in Eq. (10) we obtain

$$R^3 + PR + Q = 0 \tag{11}$$

which gives a real solution when one of the following conditions are satisfied:

$$\Delta < 0 \ or \ \Delta = 0 \ or \ \Delta >= 0 \tag{12}$$

The quantity $\Delta = (P/3)^3 + (Q/2)^2$ is the discriminant of Eq. (11) in which $P = (C/A) - (B^2/3A^2)$ and $Q = (2/27)(B/A)^3 - (BC/3A^2) + D/A$.

Fig. 2 illustrates the bifurcation curves of periodic solutions of Eq. (5). These bifurcation curves delimit two regions in which different types of solutions may exist. In the regions I, only one stable solution exists (focus). In the region II, three stationary solutions may exist (two stable focus and one saddle). Fig.3 illustrates the standard frequency response curve. In Fig.4, we show the phase plane of the slow flow system (8) corresponding to the region II.

Conclusion

In this work, we have investigated the nonlinear flexural vibrations of the fourth mode of a thin isotropic circular ring. In a first part, a softening behavior has been obtained and our analytical results have been compared with available theoretical and experimental data showing a good agreement at low amplitudes. In the second part, we have examined the periodic response and bifurcation of the fourth mode. Domain of existence of solutions has been determined using the slow flow equations. It was shown that the fourth mode of the circular ring can have one or two possible stable periodic vibrations near the 1:1 resonance.The first simple model without in-plane inertia can be adopted for the forced case analysis.

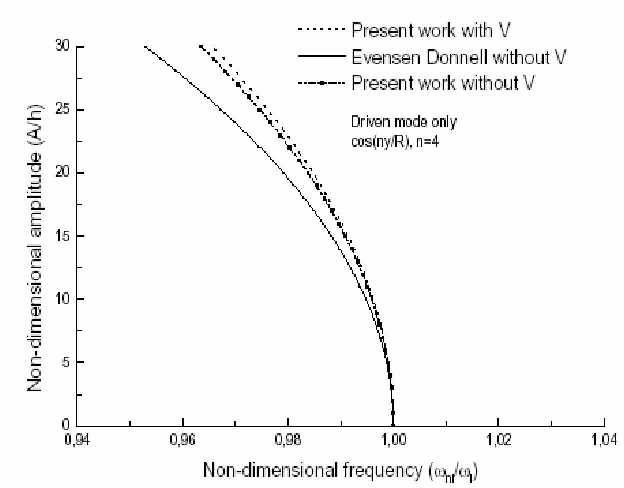

Fig.1 Backbone curves for ε =4.210^{-4}.

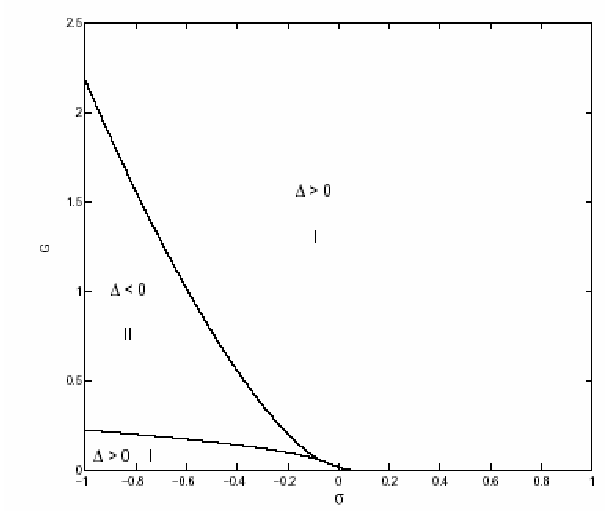

Fig.2. Bifurcation curves of periodic solution in the ($\tilde{G}$,σ) plane for $\tilde{\mu}$ =0.08 and β=1.

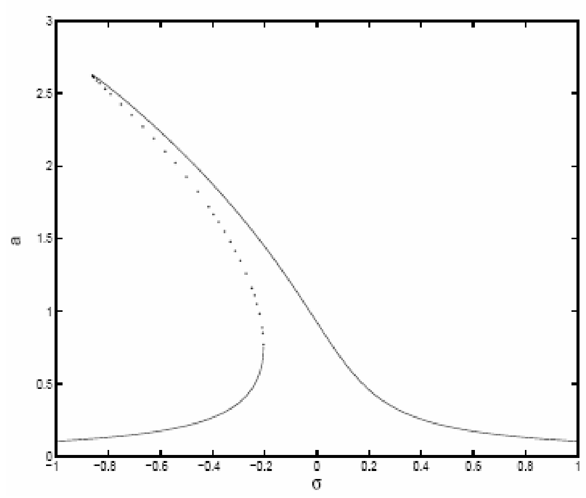

Fig.3 Amplitude frequency curve for $\tilde{\mu} = 0.08, \beta = 1$ and $\tilde{G} = 0.21$

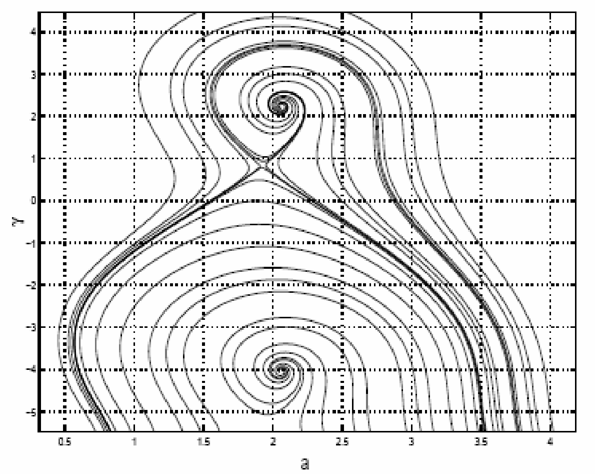

Fig. 4 The phase plane of the slow flow Eq.(8) in region II of Fig. 2 for $\tilde{\mu} = 0.08, \beta = 1, \tilde{G} = 0.21$ and $\sigma = -0.5$

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