NON-LINEAR FREE VIBRATIONS OF SIMPLY SUPPORTED CIRCULAR CYLINDRICAL SHELLS

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Introduction

It is of great technical importance to investigate the dynamic characteristics of thin shell-type structures with some complicating parameters because of their use as basic elements in modern light-weight structures. However, the studies of large vibrations amplitudes of shells involving geometrical non-linearity require efficient non-linear procedures. Very often, in the non-linear vibration analysis, an elastic system is reduced to a discrete system with one or more degrees of freedom by a suitable discretisation procedure, and a set of non-linear ordinary differential equations in time is obtained [1-5]. Such equations are usually solved by the Harmonic Balance Method (Ritz and Galerkin Averaging Method), Perturbation Method or by Multiple Scale Method [6]. In the present work, an investigation of the free non-linear flexural vibrations of simply supported isotropic circular cylindrical shells is carried out. Using a general formula for the transverse displacement, a multi-mode approach based on Lagrange's equation and the harmonic balance method is established. For testing and validating the model, the single mode approach is considered in non-linear free vibrations case.

Mathematical formulation

A circular cylindrical shell is considered, having the following characteristics: h thickness, R radius, L length.

Total strain and kinetic energy expressions

In the present paper a simplification has been adopted in the above expressions, based on some observations deduced from the analysis of the membrane and flexural strain results established in previous linear works [7], according to which the membrane strain energy may be neglected for higher circumferential modes. The total strain energy expression, V based on Donnell's non-linear shell theory is given by:

$$V = \frac{D}{2}\int_S [(\frac{\partial^2 W}{\partial x^2})^2 + (\frac{\partial^2 W}{\partial y^2})^2 + 2\nu(\frac{\partial^2 W}{\partial x^2})(\frac{\partial^2 W}{\partial y^2}) + 2(1-\nu)(\frac{\partial^2 W}{\partial x \partial y})^2]dS \quad (1)$$

In which, dS=dx dy is the elementary surface of the shell; with $x\in[0, L]$ and $y\in[0, 2\pi R]$, D is the flexural rigidity $D=Eh^3/12\,(1-\nu^2)$, E is Young's modulus, ν is Poisson's ratio.

If ρ is the shell mass density and the effects of rotary inertia and in-plane inertia are neglected, the kinetic energy expression is:

$$T = \frac{\rho h}{2}\int_S [(\frac{\partial W}{\partial t})^2]dS \quad (2)$$

Choice of the basic functions for the transverse displacement expansion

In most of the literature, the choice of an adequate expansion for transverse displacement is usually not based on a systematic indicator or a procedure, except the continuity and periodicity conditions, which may be imposed by the shell-type configuration. Based on these considerations, the transverse displacement has been written in the present model as:

$$W(x,y,t) = a_k \sin(\frac{i\pi x}{L})\cos(\frac{jy}{R})\sin\omega t - \frac{a_k a_l}{2R}\sin(\frac{k\pi x}{L}) l^2 \sin^2 \omega t \quad (3)$$

In which the usual summation convention is used, according to which summation is made for the repeated indices k and l over the range [1,..,n]. ($l^2/2R$) is the axisymmetric term coefficient. The basic functions are chosen in this work as x & y functions satisfying the simply supported boundary conditions on the normal displacement W, i.e:

W= 0 and $M_x = -D[(\frac{\partial^2 W}{\partial x^2}) + \nu(\frac{\partial^2 W}{\partial y^2})] = 0$ at x =0,L

Where M_x is the bending moment per unit length.

Discretization of the total strain and kinetic energy expressions*:*

The discretized expressions for the total strain and kinetic energies are given as:

$$V = \frac{1}{2} q_i q_j k_{ij} + \frac{1}{2} q_i q_j q_k q_l k_{ijkl} \quad (4)$$

$$T = \frac{1}{2}\dot{q}_i \dot{q}_j M_{ij} + \frac{1}{2}\dot{q}_i \dot{q}_j q_k q_l M_{ijkl} \quad (5)$$

In which K_{ij} and K_{ijkl} are the linear rigidity tensor and the fourth order non-linear rigidity tensors respectively. M_{ij} and M_{ijkl} are the mass and fourth non-linearity tensors due to the axisymmetric terms.,

Governing equations for free vibrations

Substituting the discretized expressions (4) and (5) for the kinetic and total strain energies in Lagrange's equations, leads to the following set of non-linear partial differential equations in matrix form:

$$-\omega^2([M]+\frac{1}{2}[M_{nl}(A)])\{A\}+([K]+\frac{3}{2}[K_{nl}(A)])\{A\}=\{0\} \quad (6)$$

Where $\{A\}$ is the column vector $\{A\}^t=[a_1,a_2,\ldots..a_n]$.

Non-dimensional formulation

To simplify the analysis and the numerical treatment of the set of non-linear algebraic equations, non-dimensional formulation has been considered by putting the spatial displacement functions as:

$$W(x,y,t)=hw^*(x^*,y^*,t) \quad (7)$$

$$w^*=\frac{a_k}{h}w_k(x,y)\sin\omega t-(\frac{a_l a_k}{h^2})(\frac{\beta}{2})g_k(x)h_k(y)\sin^2\omega t \quad (8)$$

The geometrical parameters of the shell are defined by:

$$\gamma=R/L,\ \beta=h/R \quad \text{and} \quad x^*=x/L,\ y^*=y/R \quad (9\text{-}13)$$

are the longitudinal and circumferential non-dimensional co-ordinates, and the non-dimensional elementary surface $ds^*=dx^*dy^*$ with $x^*\in[0,1]$ and $y^*\in[0,2\pi]$. The non-dimensional tensors are given by:

$$K_{ij}^*=\frac{\beta^2}{12}\int_{s^*}\left\{\begin{array}{l}\gamma^4(\frac{\partial^2 w_i^*}{\partial x^{*2}})(\frac{\partial^2 w_j^*}{\partial x^{*2}})+(\frac{\partial^2 w_i^*}{\partial y^{*2}})(\frac{\partial^2 w_j^*}{\partial y^{*2}})+\\ 2\nu\gamma^2(\frac{\partial^2 w_i^*}{\partial x^{*2}})(\frac{\partial^2 w_j^*}{\partial y^{*2}})\end{array}\right\}ds^*+$$

$$\frac{\beta^2}{12}\int_{s^*}\left\{2(1-\nu)\gamma^2(\frac{\partial^2 w_i^*}{\partial x^*\partial y^*})(\frac{\partial^2 w_j^*}{\partial x^*\partial y^*})\right\}ds^* \quad (14)$$

$$M_{ij}^*=\int_{s^*}w_i^*w_j^*ds^*;\ M_{ijkl}^*=\beta^2\int_{s^*}g_i^*g_j^*h_k^*h_l^*ds^* \quad (15\text{-}16)$$

$$K_{ijkl}^*=\frac{\beta^2\gamma^2}{48}\int_{s^*}\left\{(\frac{\partial^2 g_i^*}{\partial x^{*2}})(\frac{\partial^2 g_j^*}{\partial x^{*2}})h_k h_l\right\}ds^* \quad (17)$$

The set of non-linear equations (6) can be rewritten in non-dimensional form as:

$$\begin{array}{l}-\omega^2\frac{\rho h^3R^2}{12D}(a_iM_{ir}^*+\frac{1}{2}a_ia_ja_kM_{ijkr}^*)+\\ (a_iK_{ir}^*+\frac{3}{2}a_ia_ja_kK_{ijkr}^*)=0\end{array} \quad (18)$$

Application to the single mode approach

General Considerations concerning mode shapes determination

If one mode is considered (only the driven mode participates to free vibrations), the free vibration mode shapes for the cylinder are easily shown to be given by equation (19):

$$w_{mn}=a\sin(\frac{m\pi x}{L})\cos(\frac{ny}{R})\sin\omega t-\frac{a^2}{2R}\sin(\frac{m\pi x}{L})\sin^2\omega t$$

which becomes in non-dimensional formulation:

$$w^*_{mn}=a\sin(m\pi x^*)\cos(ny^*)\sin\omega t-(\frac{\beta a^2}{2})\sin(m\pi x^*)\sin^2\omega t \quad (20)$$

Linear and non-linear frequency expressions

Equations (18) written in the case of a single mode approach lead to a formula for the non-linear frequency parameter:

$$(\frac{\omega^*_{nl}}{\omega^*_l})^2=\frac{1+\frac{3}{4}\frac{\xi^4\varepsilon}{(1+\xi^2)^2}a^2}{1+\varepsilon a^2} \quad (21)$$

Where the parameters ε and ξ, are defined by: $\varepsilon=\beta^2n^4$; $\xi=\frac{m\pi\gamma}{n}$;

Numerical results

Numerical results, based on equations (21), have been computed for some classical shell cases examined recently or in old literature. Comparisons of backbone curves obtained by Ganapathi & Varadan [4], Amabili & Paidoussis [5] and the present results show a good concordance, as can be seen in table 1.
The second case analysed here corresponded to the shell experimentally tested by Olson [8], and theoretically examined by Evensen [1], Ganapathi & Varadan [4], Amabili & Paidoussis [5].

In order to make qualitative comparisons, corresponding to the above-mentioned cases, values read on graphs have been summarised in tables (3, 4), and compared with the present ones. It can be seen that in all cases, the differences do not exceed 0.15%. This allows is to conclude that the simple formulae obtained here, i.e. formulae (21), can be reasonably used to estimate simply the non-linear response of simply supported shells at large vibration amplitudes.

a/h	Present work ω/ω_l	ref [4] ω/ω_l	ref [3] ω/ω_l	ref [1] ω/ω_l	Standard Deviation
0	1	1	1	1	0
0.1	1	1.00001	1	1.00001	5.77310^{-6}
0.2	0.99981	0.99982	0.99992	0.99992	5.966810^{-5}
0.3	0.99959	0.99959	0.99982	0.99981	1.282610^{-4}
0.4	0.99932	0.99936	0.99965	0.99965	1.778510^{-4}
0.5	0.99895	0.99912	0.99933	0.99948	2.335310^{-4}
0.6	0.99850	0.99885	0.99907	0.99923	3.163810^{-4}

Table 2: Quantitative comparison between results obtained analytically and results obtained by several authors with various methods; Shell tested by Chen & Babcok (n=6, ε=0.26179, ζ=0.007906, ν=0.31).(Backbones curves)

a/h	Present work ω/ω_l	ref [9] ω/ω_l	ref [3] ω/ω_l	ref [5] ω/ω_l	Standard Deviation
0	1	1	1	1	0
0.2	0.99995	0.99992	0.99982	0.99985	6.098010^{-5}
0.4	0.99977	0.99960	0.99950	0.99950	1.259810^{-4}
0.6	0.99946	0.99936	0.99896	0.99892	2.73610^{-4}
0.8	0.99898	0.99805	0.99834	0.99814	4.194410^{-4}
1	0.99850	0.99880	0.99766	0.99706	7.924910^{-4}
1.2	0.99779	0.99852	0.99702	0.99585	1.143210^{-3}
1.4	0.99694	0.99814	0.99637	0.99436	1.577210^{-3}
1.5	0.99653	0.99800	0.99600	0.99355	1.850610^{-3}
1.6	0.99602		0.99556		3.309210^{-4}
1.8	0.99508		0.99477		2.135410^{-4}
2	0.99396		0.99389		4.808310^{-5}
2.2	0.99268		0.99301		2.368810^{-4}
2.4	0.99126		0.99207		5.678010^{-4}

Table 2: Quantitative comparison between results obtained analytically and results obtained by several authors with various methods; Shell tested by Olson (n=10, ε=0.003025, ζ=0.1635, ν=0.365).(Backbones curves)

Length (m)	0.2
Radius (m)	0.1
Thickness (m)	0.000247
Young’s modulus(Pa)	71.0210^{9}
Poisson’s ratio	0.31
Density(Kg/m^3)	2796
Mode (n, m)	(6,1)

Table3: Geometrical and material characteristics of the circular cylindrical shell considered by Chen and Babckok [3]

Length (m)	0.39
Radius (m)	0.203
Thickness (m)	0.1 1176 x10^{-3}
Young’s modulus (Pa)	124 x10^{9}
Poisson’s ratio	0.35
Density (Kg/m^3)	8930
Mode (n, m)	(10,1)

Table4: Geometrical and material characteristics of the circular cylindrical shell considered by Olsen [8]

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