# Proper frequencies of the cochlea and the amplification of the best frequency

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**Abstract**

The passive mechanical sight of the cochlea is considered in this work. The flow in the scala vestibuli and in the scala timpani of cochlea channels are three potential flows and the membrane is considered as two dimension shell. A full interaction between the fluids in the cochlea channels and the membrane is considered. The aim of the work is to explain the passive amplification of the basilar membrane by cochlea resonance.

**Introduction**

The ear is an advanced and very sensitive organ of the human body. One of the major tasks of the ear is to detect and analyze noises by transduction. The ear is divided into three different parts, the outer ear, the middle ear and the inner ear. In this work we study the mechanical behavior of the inner ear (the cochlea).

The cochlea is a spiraled, hollow, conical chamber of bone. Structures include, the scala vestibuli (containing perilymph), which lies superior to the cochlea duct and abuts the oval window, see figure 1. The scala tympani (containing perilymph ), which lies inferior to the scala media and terminates at the round window. The scala media (containing endolymph), which is the membraneous cochlea duct containing the organ of Corti. The helicotrema is the location where the scala tympani and the scala vestibuli merge. Reissner's membrane separates the scala vestibuli from the scala media. The basilar membrane separates the scala media from the scala tympani. The Organ of Corti is a cellular layer sitting on top of the basilar membrane. It is lined with hair sensory cells topped with hair-like structures called stereocilia. Humans can hear sounds waves with frequencies between 20 and 20,000 Hz. The three bones in the ear (malleus, incus, stapes) pass these vibrations on to the cochlea. This vibration is transmitted to the organ of Corti which contains the Hair cells and the basilar membran. The cilia (the hair) of the hair cells make contact with another membrane called the tectorial membrane. When the hair cells are excited by vibration, a nerve impulse is generated in the auditory nerve. These impulses are then sent to the brain.

In order to simplify the mathematical description of the mechanical behavior of the cochlea we consider it as a plan rather than conical chamber, figure 1. The basilar membrane, Reissner's membrane, the scala media and the organ of Corti are considered as an elastic shell. The fluid in the scala vestibuli channel and scala timpani channel are non viscous and the flow is no rotational.

The acoustic of the cochlea has been considered, among others, by Nobili and mammano (1993) and Egber de Beor (2000), Norman sieroka et al (2006) where a several model have been proposed to explain the amplification of the basilar membrane vibrations by an eventual work which is supposed to be done by the hair cells. In our work we focus on the amplification of the basilar membrane by the resonance phenomena of the cochlea seen as a passive chamber.

**Formulation of the problem**

The flow in the scala vestibuli and in the scala timpani are supposed to be irrotation flows. Then the velocities in those two organs are, respectively, the gradient of two scalar functions $\phi_1$ and $\phi_2$ . Let $W(x,z,t)$ be the displacement of the basilar membrane, $V$ is the velocity of the membrane of the oval window and $L$ is the basilar membrane length. The height of the of the channel, $h$, is considered as the unity of length. Therefore, the equations of the fluid are

$$\frac{\partial^2 \Phi_1}{\partial x^2} = \frac{1}{\frac{b\pi}{c} - w}\left(\frac{\partial \Phi_1}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\right)$$

$$\frac{\partial^2 \Phi_2}{\partial x^2} = \frac{-1}{\frac{b\pi}{c} + w}\left(\frac{\partial \Phi_2}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\right)$$

and the membrane equation is

$$\frac{\partial^2 w}{\partial t^2} + D\frac{\partial^4 w}{\partial x^4} - 2D\beta^2\frac{\partial^2 w}{\partial x^2} + D\beta^4 w = \frac{\pi\rho}{c}\left[\frac{\partial(\Phi_1 - \Phi_2)}{\partial t} + \frac{1}{2}\left({\frac{\partial \Phi_1}{\partial x}}^2 - {\frac{\partial \Phi_2}{\partial x}}^2\right)\right]$$

. As the dimension of the helicotrema chamber is small, we consider it as a point located at $x=L$. Therefore, the boundary conditions associated with the precedent equations are

$$\frac{\partial \Phi_1}{\partial x} = -\frac{\partial \Phi_2}{\partial x} = V\ ,\quad w = \frac{\partial w}{\partial x} = 0\ ,\qquad x = 0$$

$$\Phi_1 = \Phi_2\ ,\quad \frac{\partial \Phi_1}{\partial x} = -\frac{\partial \Phi_2}{\partial x}\ ,\quad w = \frac{\partial w}{\partial x} = 0\ ,\qquad x = L$$

**Linear equations**

In order to find the resonance frequency, we neglect the no linear terms to find

$$\left[\frac{\partial^2}{\partial t^2} + D\left(\frac{\partial^2}{\partial x^2} - \beta^2\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} = 2\rho_r\frac{\partial^2 \phi}{\partial t^2}$$

with the boundary conditions

$$\frac{\partial \phi}{\partial x} = V,\quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^3 \phi}{\partial x^3} = 0\ ;\qquad x = 0$$

$$\phi = 0,\quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^3 \phi}{\partial x^3} = 0\ ;\qquad x = L$$

where $\phi = \phi_1 - \phi_2$

**Modes propers**

We seek a solution in the form of normal mode, namely

$$\phi = \tilde{\phi} e^{i\omega t + ikx}$$

The linear equation reads

$$[\frac{\partial^2}{\partial t^2} + D(\frac{\partial^2}{\partial x^2} - \beta^2)^2]\frac{\partial^2 \phi}{\partial x^2} = -2\omega^2 \rho_r \phi$$

and the condition are

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^3 \phi}{\partial x^3} = 0 \ ; \qquad x = 0$$

$$\phi = 0, \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^3 \phi}{\partial x^3} = 0 \ ; \qquad x = L$$

which leads to the dispersion equation, that is

$$D(k^2 + \beta^2)^2 k^2 - (k^2 + 2\rho_r)\omega^2 = 0$$

**No linear dynamic equation**

Solving in formal way the equation relating to $\phi_1$ and $\phi_2$ and eliminating $\phi_1$ and $\phi_2$ from the membrane equation leads to the following nonlinear equation

$$\frac{\partial^2 w}{\partial t^2} + D\frac{\partial^4 w}{\partial x^4} - 2D\beta^2\frac{\partial^2 w}{\partial x^2} + D\beta^4 w =$$
$$\frac{\pi\rho}{c}\{\int_x^L [-\frac{\partial w(z)}{\partial t}(\frac{1}{(w(z)-\gamma)^2} - \frac{1}{(w(z)+\gamma)^2})(\int_0^z \frac{\partial w(\alpha)}{\partial t}d\alpha + \gamma V)]dz$$
$$+\frac{1}{2}(\frac{1}{(w-\gamma)^2} - \frac{1}{(w+\gamma)^2})[(\int_0^x \frac{\partial w(\alpha)}{\partial t}d\alpha)^2 + (\gamma V)^2]\}$$

which describes the membrane motion, with the boundary conditions

$$w = \frac{\partial w}{\partial x} = 0 \ , \qquad x = 0 \qquad ; \qquad w = \frac{\partial w}{\partial x} = 0 \ , \qquad x = L$$

**Results**

In the numerical results, we supposed that the frequency is real and we look for the spatial eigenvalues. The complex temporal eigenvalues is ignored in this work. As the dispersion equation is of six order, there are six spatial eigenvalues for each frequency. Figure 2 shows the two real spatial modes versus the frequency, they are equal in absolute values and opposite in sign. Figures 3 shows two complexes modes versus the frequency. The two modes are complex number of equal real part at low frequency and real but different at high frequency. The two modes coalesce to form one eigenvalues which form a root of second order of the dispersion relation. Figure 4 shows two other complexes modes forming a coalescence point like the two modes shown in figure 3. Thus, the dispersion equation has two doubles roots for the frequency of the order of 1200 Hz for the parameters considered in this figures. This explain way the cochlea is sensitive to a particular frequency commonly called the best frequency. In fact, taking account the double roots of the dispersion equation we can show easily that amplification of the basilar membrane is more important in vicinity of this frequency. In fact, let $\omega_p$ be the frequency of the acoustic wave and $\omega_i$ i=1,2… is the proper frequencies of the cochlea, then it can be shown that when $\omega_p$ goes to $\omega_i$ the solution goes to infinity as $|\omega_p - \omega_i|^{-1}$ when $\omega_p$ is a simple root of the dispersion equation while the solution goes to infinity as $|\omega_p - \omega_i|^{-2}$ in the coalescence point. Moreover, the eigen vector is of the form $xe^{i\alpha x}$ , $\alpha$ is the proper wave number. Thus, for a mammalian cochlea the eigenvector is 6 to 10 higher than the other eigenvecrot at the basilar membrane apex, $0 < x < x_{max}$ , $(6 < x_{max} < 10)$. This results is consistent with the experimental observation of the vibration of the basilar membrane at its best frequency. Figure 5 shows the best frequency, where the double root of the dispersion euqation occurs, versus the mass density ratio of the basilar membrane and for some aspect ratios.

**Conclusion**

In this paper we formulated the cochlea response to an incoming acoustic wave taking into account the full interaction between the fluid in the scala vestibuli and the scala media and the basilar membrane. We find all the resonance frequency and wave numbers of the cochlea. We found that the dispersion relation has two double routs. Then, we explained why the cochlea has a best frequency and we established the multiplication factors of the basilar membranes vibration at this best frequency. The best frequency found in our computation is in fair agreement with the experimental finding. By formal elimination of the potential flow involved in the force terms we established the nonlinear dynamic equation of the basilar membranes which involves only the displacement of the basilar membrane. Such an equation is very helpful in studding the compressive character at high frequency of the cochlea.

**References**

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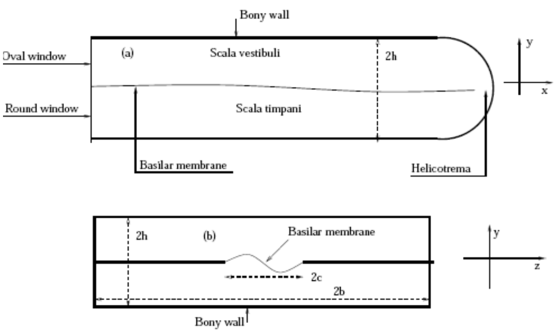



Figure1: Cartoon of the cochlea

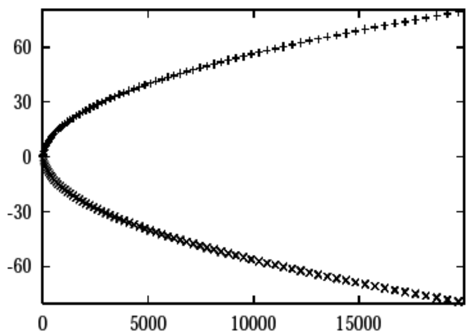



Figure 2: Two spatial real eigenvalues (ordinate) of the system versus the frequency (Abscissa).

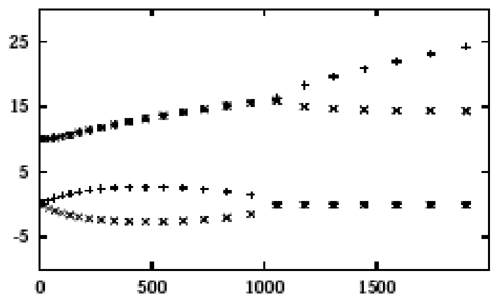



Figure 3: Two spatial eigenvalues (ordinate) versus the frequency (abscissa). The figure shows the two modes which coalesce to form a root of second order of the dispersion relation.

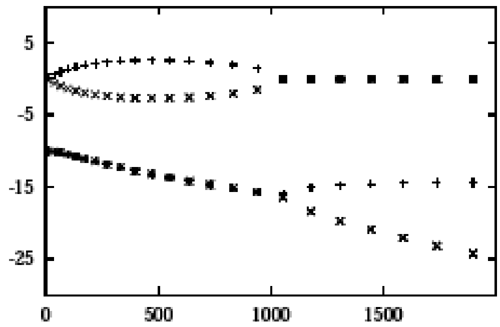
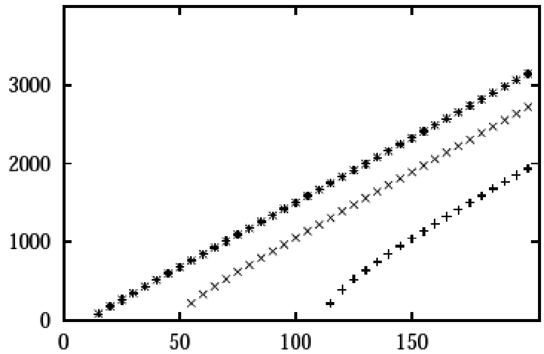



Figure 4: Two other spatial eigenvalues (ordinate) versus the frequency (abscissa). The shows the two others modes which coalesce to form a root of second order of the dispersion relation.




The frequency where double root occurs versus the mass density ratio $\rho_r$ for some values of $\beta$ which equals from above (2◊,15,10,5).