NUMERICAL SOLUTION OF HEAT TRANSIENT CONDUCTION IN ANISOTROPIC CYLINDER

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Abstract: This study presents numerical solution of transient heat conduction in anisotropic cylinder, which subjected to prescribed temperature on two sections and cooling at the side surface. Investigations on anisotropic heat conduction problems are tedious due to the presence of the cross-derivatives in the governing differential equation. A linear coordinate transformation is introduced to simplify the problem and reduce the anisotropic equation into the corresponding isotropic ones without complicating the boundary conditions and with complicate geometry. The isotropic equation combined with the boundary conditions is integrated using the alternating-direction implicit finite-difference method (ADI). Inverse transformation provides profile of temperature in anisotropic cylinder. The numerical code is validated with an orthotropic steady state solution available in the literature subject to the same type of boundary conditions. The paper discusses the effect of anisotropy (k_{rz}) and radial Biot number (B_{Ir}) on unsteady heat transfer inside anisotropic middle which their principal thermal conductivity can be equal or different for symmetrical boundary conditions (\theta_L=\theta_R).

Keywords: Anisotropic, transient heat conduction, linear coordinates transformation, ADI

1. Introduction

In recent years the study on anisotropic materials had been of great interest in applied science and engineering. In heat conduction science, anisotropic materials have a specific thermal behavior because their thermal conductivity varies with direction unlike the isotropic material which the thermal conductivity is constant. Therefore the simulation of heat conduction in this type of materials is of great importance. One of the mathematical difficulties in anisotropic heat conduction is the presence of cross-derivatives in the governing differential equation. When the cross-derivatives are absent from the heat conduction equation, as in the case of orthotropic solids, the analysis is significantly simplified and solutions for such cases have been reported in several references [1-2]. In recent years, several works have appeared in the literature on the solution of steady state heat conduction problems in anisotropic media [3-5]. For transient heat conduction, Osizick and Shouman [7] found a general solution of anisotropic heat conduction for an infinitely long solid or hollow cylinder. This form of integrals solution is valid only for infinite geometry.

The present paper presents a numerical solution for transient conductive heat transfer in finite anisotropic solid cylinder. We focus on axisymmetric heat transfer in anisotropic cylinder by considering heat conduction in the longitudinal and radial directions (r,z). The numerical solution can be used to simulate the profile of temperature in anisotropic media. The object is to study effect of mixed conductivity and radial Biot number on heat conduction inside anisotropic solid cylinder.

2. Statement of the problem and mathematical formulation:

Consider an anisotropic cylinder of length L and radius r_0 subjected to boundary conditions as shown in fig.1 and 2:

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Configurations studied.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Configuration studied in the plane (r,z).}
\end{figure}

We assume that heat generation is absent, the physical and thermal properties of the medium are constant and the thermal conductivity tensor is symmetric and positive definite. The governing partial differential equation for the unsteady state heat conduction problem in a two dimensional coordinates system for anisotropic cylindrical solid is given by [6]:

\begin{equation}
\frac{\partial^2 T}{\partial r^2} + \frac{k_{rr}}{r} \frac{\partial T}{\partial r} + \frac{k_{rz}}{r} \frac{\partial^2 T}{\partial z^2} + 2k_{rz} \frac{\partial}{\partial z} + k_{zz} \frac{\partial^2 T}{\partial z^2} = \frac{\partial \rho c_p T}{\partial t}.
\end{equation}

The corresponding heat fluxes are given as:

\begin{align}
q_r &= -k_{rr} \frac{\partial T}{\partial r} - k_{rz} \frac{\partial T}{\partial z} \label{eq:q_r} \\
q_z &= -k_{rz} \frac{\partial T}{\partial r} - k_{zz} \frac{\partial T}{\partial z} \label{eq:q_z}
\end{align}

In order to eliminate the cross-derivative and reduce the governing equation (1) to an isotropic form, a special linear coordinate transformation according [5] is introduced as:

\begin{equation}
\begin{bmatrix}
R \\
Z
\end{bmatrix} = \begin{bmatrix}
a & 0 \\
c & 1
\end{bmatrix} \begin{bmatrix}
r \\
z
\end{bmatrix}
\end{equation}

With \(a = \sqrt{\frac{k_{rr}k_{zz}-k_{rz}^2}{k_{rr}}}\) and \(c = -\frac{k_{rz}}{k_{rr}}\)
Equation (1) can be rewritten in new coordinate system (R,Z) as:

\[
\begin{align*}
  k \frac{\partial T}{\partial R} + k \frac{\partial^2 T}{\partial R^2} + k \frac{\partial^2 T}{\partial Z^2} &= \rho c_p \frac{\partial T}{\partial t} \\
  \text{where} \quad k &= a^2, k_{rr} \\
  \end{align*}
\]

(4)

The relationships between the heat fluxes in the two coordinate systems are given by:

\[
\begin{align*}
  q_r &= -a.k_{rr} \frac{\partial T}{\partial R} \\
  q_z &= -(a.k_{rz}) \frac{\partial T}{\partial R} - k \frac{\partial T}{\partial Z} \\
  \end{align*}
\]

(5) (6)

And its appropriate geometry is as shown below:

![Fig.3. Geometry of the medium studied in the virtual space.](image)

The determination of the profile of temperature \( \theta \) in the virtual dimensionless space \((R^*, Z^*, t^*)\) allows by inverse transformation to deduce the evolution of the temperature \( \theta \) in the real space \( R, Z \):

\[
\begin{align*}
  \theta(R, Z) &= \frac{\theta(R, Z)}{\theta_{ref}} \quad R = R^* R_0 \quad Z = Z^* Z_0 \\
  \end{align*}
\]

3. Numerical solution and validation

The numerical method retained for this study to solve the general heat-conduction problem is the alternating-direction implicit finite-difference (ADI).

Equation (7) is solved simultaneously for the three media of the integration domain: Both triangular bands besides and the virtual isotropic middle. The profiles of temperature are linked by the boundary conditions along the oblique left and right straight lines.

In order to validate the numerical code, we consider analytical solution available in literature, which have subjected to the same boundary conditions as our problem. Thus, the numerical solution is tested with steady-state analytical solution of temperature in orthotropic media expressed by [6]:

\[
\begin{align*}
  \theta(R^*, Z^*) &= 2.0 \sum_{n=1}^{\infty} \frac{\lambda_n J_0(\lambda_n R_0) J_1(\lambda_n Z_0)}{J_0^2(\lambda_n R_0)(1/4 + \lambda_n^2)} \times \\
  &\times \frac{\sin(\lambda_n \varepsilon (Z^* - Z_0))}{(2 \lambda_n^2) \sinh(2 \lambda_n A)} \\
  \end{align*}
\]

(10)

Where \( \varepsilon = \frac{k_{rr}}{k_{xx}} \) and \( A = \frac{1}{r_0} \) is the form factor.

The eigenvalue \( \lambda_n \) for equation (10) is obtained by solving eigenvalues equations (11):

\[
\lambda_n J_1(\lambda_n) - B_{ir} J_0(\lambda_n) = 0 \\
\]

(11)

To do this comparison we decreased the mixed conductivity and we take \( k_{rr} \neq 1 \). The comparison is making for two numbers values of radial Biot number: 1 and 5:

![Fig.5. Temperature profiles. θ₀ = 0, θ₁ = 1 and θ₂ = 0](image)
As fig. 5 shows, numerical results agree completely with analytical solution.

Thermal behavior of anisotropic medium is studied for various values of cross thermal conductivity $k_{cr}$, radial Biot number $B_{tr}$, principal conductivity $\frac{k_{xx}}{k_{rr}} = 1$ and $\frac{k_{zz}}{k_{rr}} \neq 1$ for symmetrical configuration $\theta_L = \theta_R = 1$, $\theta_t = \theta_t = 0$. Result are reported in Fig. 6

**CONCLUSION**

In this paper we have developed a numerical solution of transient heat conduction in anisotropic cylinder. The obtained results for symmetrical configuration in terms of boundary conditions, showed the effect of mixed-conductivity and radial Biot number in profile of temperature inside anisotropic middle

**References**


Fig. 6. Temperature profiles in anisotropic cylinder under the effect of mixed-conductivity, principal conductivity and radial Biot number.