AXISYMMETRIC SHAPE OF GAS BUBBLES IN A UNIFORM FLOW

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Abstract:
This paper investigates axisymmetric shapes of an incompressible inviscid fluid bubble in a uniform flow. Approximate solutions are constructed for the governing nonlinear boundary value problem using domain perturbation method. Bifurcation and turning points are located by applying a special type of Hermite-Padé approximation.

Keywords: Domain perturbation method, Axisymmetric shape, Hermite–Padé approximation, Bifurcation study.

1. Introduction
The rupture of liquid drops by a gas flow is of great importance in engineering and technology, including spray painting, agriculture (drip irrigation, pesticide drift), atomization etc. In combustion theory, models of fuel droplets are used to describe droplet vaporization in a high-temperature environment in devices such as diesel engines, gas turbines, furnaces and rocket motors [1].

The situation of a single drop subjected to aerodynamic forces due to slip velocity between the drop and the surrounding gas can also simulate a raindrop falling through the atmosphere [2]. Since the early 1950's many theoretical and experimental works has been performed on the study of the deformation of a liquid drop or gas bubble in irrotational flow. The analysis of the drop shapes goes back to Davies and Taylor [3].

The aim of this paper is the computation of axisymmetric shapes of a deformable bubble in a uniform flow. We use a domain perturbation method which consists in taking as principal unknown the transformation field from the initial known position of the bubble to the unknown position depending on the Weber number. Velocity and pressure solution for the flow and the shape of the bubble are determined in terms of series expansion in power of the Weber number. Bifurcation and turning points are located by applying a special type of Hermite Padé approximation. This technique has been recently used in [4] in order to determine the shape of rotating drops.

2. Mathematical formulation
We consider a steady flow caused by rectilinear motion of a bubble with constant velocity \( V_\infty \). Viscosity and gravity are both neglected. The flow is assumed to be irrotational and incompressible. The shape of the bubble is governed by the balance between the dynamic of the pressure distribution and the surface tension. The problem will be set in a reference frame \( Oxyz \) with the origin attached to the centre bubble. We assume without loss of generality, that \( V_\infty \) is parallel to the coordinate axis \( Oz \). (cf. figure 1).

Let \( \mathbf{v} = V_\infty + \nabla \mathbf{u} \), be the velocity of the flow and \( u \) the potential due to the presence of the bubble, with \( u \to 0 \), and \( \nabla u \to 0 \) at infinity. The incompressibility condition \( \nabla \cdot \mathbf{v} = 0 \) and the slip condition at the boundary of the bubble give the following dimensionless governing equations for the flow in the unknown domain \( \Omega \) [5].

\[
\begin{align*}
\Delta \tilde{u} &= 0 & \text{in } \Omega^e \\
\frac{\partial \tilde{u}}{\partial n} &= -e_z \cdot \mathbf{n} & \text{on } \partial \Omega
\end{align*}
\]

\[
\lambda \left[ \frac{1}{2} |\nabla \tilde{u}|^2 - \nabla \tilde{u} \cdot e_z \right] - \tilde{C} = k
\]

\( \lambda = \frac{R \rho V_\infty^2}{\sigma} \) is the Weber number which is the ratio of inertial forces to interfacial ones. \( k \) is the reference pressure, an unknown constant depending on \( \lambda \) can be determined by the volume constraint: \( V = \frac{4}{3} \pi R^3 = V_0 \) (3)

If \( \lambda = 0 \), the equilibrium shape is a sphere \( \Omega_0 \) of radius \( R \), taken as a fixed reference domain.

If \( \lambda \neq 0 \), the equilibrium configuration is \( \Omega \). So \( \lambda \) will be considered as an expansion parameter for this problem.

We consider as principal unknown the transformation field \( x = T(X, \lambda) \) from the initial known position \( \Omega_0 \) to the unknown one. A spherical coordinate system \((r, \theta, \varphi)\) has to be assumed, with the origin at the centre of the drop and with the \( z \) axis as the rotation axis. The form of the transformation field is chosen as:

\[
T = r f(\theta, \lambda) e_r
\]

If \( V_\infty \) is changed to - \( V_\infty \) the configuration is unchanged, then by symmetry:

\[
f(\pi - \theta, \lambda) = f(\theta, \lambda)
\]

Equation (1.1) can be written:

\[
F_1 r + \frac{2}{r} F_{10} + \frac{1}{r} F_{20} + \frac{\cos \theta}{r \sin \theta} F_2 = 0 \text{ for } r > 1
\]

where
The subscript notation denotes partial differentiation.

The boundary condition (1.2) becomes, for \( r = 1 \):

\[
\left( f + f_\theta \right) \left( u_r - \cos \theta \right) - f_\theta \left( u_\theta - \sin \theta \right) = 0
\]

(7)

While relation (2) takes the following form:

\[
\lambda \left[ \frac{1}{2} \left( u_r^2 + F_r^2 \right) - u_r \cos \theta + F_r \left( \sin \theta + \frac{f_\theta \mu_r}{f} \right) \right] - \left( C + k \right) f^2 = 0
\]

for \( r = 1 \),

(8)

with: \( F_r(r, \theta) = u_\theta - \frac{f_\theta u_r}{f} \). In term of interface equation, the curvature \( C \) can be written:

\[
C = \frac{f \theta \cos \theta - f \sin \theta}{f \sin \left( f^2 + f_\theta^2 \right)^{1/2}}.
\]

Then \( u, k \) and \( f \) will be sought as series of \( \delta \):

\[
\begin{align*}
\lambda(r, \theta) &= \sum_{n=0} a_n u_n(r, \theta) \\
\lambda(\theta) &= \sum_{n=0} a_n g_n(\theta) \\
k &= \sum_{n=0} \lambda^n h_n
\end{align*}
\]

(9)

Using a computer symbolic algebra package, we obtained the first 24 terms of the above solution series.

3. Bifurcation study

The main tool of this paper is a simple technique of series summation based on a special type of Hermite-Padé approximation technique and may be described as follows.

Let us suppose that the partial sum

\[
U_{\alpha}(\delta) = \sum_{m=0} a_m \delta^m + O(\delta^{m+1})
\]

as \( \delta \to 0 \)

(10)

is given, we construct a multivariate series expression of the form:

\[
F_{\mu}(\delta, U_{N-1}) = 1 + A_{N}(\delta) U^{(1)} + A_{2N}(\delta) U^{(2)} + A_{3N}(\delta) U^{(3)}
\]

whereby, we substitute in equation (9):

\[
U^{(1)} = U, \quad U^{(2)} = U^2, \quad U^{(3)} = U^3\text{ for cubic algebraic approximant and}
\]

\[
U^{(1)} = U, \quad U^{(2)} = \frac{dU}{d\delta}, \quad U^{(3)} = \frac{d^2U}{d\delta^2}\text{ for second order differential approximant, such that}
\]

\[
A_{\alpha}(\delta) = \sum_{m=1}^{M} a_m \delta^m \text{ and } F_{\mu}(\delta, U) = O(\delta^{m+1})\text{ as } \delta \to 0
\]

(11)

with \( M \geq 1 \), \( i = 1, 2, 3 \). The unknowns coefficients \( a_{ij} \) depend only on the \( N \) given coefficients \( a_0 \) and we shall take \( N = 3(2 + M) \) so that the number of equations equals the number of unknowns. For details on the above procedure, interested readers can see [5]. Using the above procedure on the solution series \( U = R_{max} = f(0, \lambda) \), we obtained the results as shown in table below:

<table>
<thead>
<tr>
<th>( M )</th>
<th>( N )</th>
<th>( \lambda_c N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>15</td>
<td>1.273698428</td>
</tr>
<tr>
<td>5</td>
<td>18</td>
<td>1.273876133</td>
</tr>
<tr>
<td>6</td>
<td>21</td>
<td>1.273898428</td>
</tr>
<tr>
<td>7</td>
<td>24</td>
<td>1.273890627</td>
</tr>
</tbody>
</table>

Table: Computations showing the procedure rapid convergence and bifurcation point of the aspect ratio \( e(\lambda) \)

One can see that the direct calculation gives the estimation \( \lambda_c = 1.2737 \).

4. Conclusions

Axisymmetric equilibrium shapes of bubbles or liquid drops in a uniform flow of constant velocity are found using a domain perturbation method. A bifurcation study by analytic continuation of a power series is performed using a special Hermite-Padé approximant.

5. References


